Introduction to Geometric Nonlinear Control; Linearization, Observability, Decoupling

Witold Respondek*

Laboratoire de Mathématiques, INSA de Rouen, France

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LNS028004

*wresp@insa-rouen.fr

Abstract

These notes are devoted to the problems of linearization, observability, and decoupling of nonlinear control systems. Together with notes of Bronislaw Jakubczyk in the same volume, they form an introduction to geometric methods in nonlinear control theory. In the first part we discuss equivalence of control systems. We consider various aspects of the problem: state-space and feedback equivalence, local and global equivalence, equivalence to linear and partially linear systems. In the second part we present the notion of observability and give a geometric rank condition for local observability and an algebraic characterization of local observability. We discuss uniform observability, decompositions of nonobservable systems, and properties of generic observable systems. In the third part we introduce the notion of invariant distributions and discuss disturbance decoupling and input-output decoupling. Many concepts and results are illustrated with examples.

Contents

1 Introduction

These notes, together with notes of Jakubczyk [26] of the same volume, form an introduction to geometric nonlinear control. Section 2 has an elementary and introductory character. We formulate the problem of feedback linearization, show why the concept of Lie bracket appears naturally, and give necessary and sufficient conditions for feedback linearization in the single-input case. In Section 3 we introduce two concepts of equivalence of control systems: state space equivalence and feedback equivalence. We also state a result that any nonlinear control system is (locally) determined by iterated Lie brackets of vector fields corresponding to constant controls. In Section 4 we discuss various aspects of the feedback linearization problem. In particular, we consider the multi-input as well as non control-affine systems and the problems of global feedback linearization, restricted feedback linearization, and partial linearization. Section 5 is concerned with the concept of observability. We introduce observability rank condition and then discuss Kalman-like decomposition of nonlinear non observable systems, uniform observability, and generic properties of observable systems. Finally, in Section 6 we introduce te concept of invariant and controlled invariant distributions and, based on it, discuss solutions to the disturbance decoupling and input-output decoupling problems.

We do not provide proofs of the presented results and send the reader to the literature on geometric control theory (see the list of references) and, in particular, to monographs [18], [23], [29], [37]. As a small "recompense", we illustrate many notions, concepts, and results by simple, mainly mechanical, examples.

2 Feedback linearization: an introduction

The aim of this preliminary section is to introduce the concept of feedback linearization and a fundamental geometric tool of nonlinear control theory, which is the Lie bracket. Feedback linearization is a procedure of transforming a nonlinear system into the simplest possible form, that is, into a linear system. Necessary and sufficient conditions for this to be possible will be expressed using the notion of Lie bracket, which is omnipresent in very many nonlinear control problems.

The problem of feedback linearization is to transform the nonlinear control system

$$\dot{x} = f(x, u)$$

into a linear system of the form

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

via a diffeomorhism

$$(\tilde{x}, \tilde{u}) = (\Phi(x), \Psi(x, u)),$$

called feedback transformation. We will start with an introductory example.

Example 2.1 Consider a nonlinear pendulum (rigid one-link manipulator) consisting of a mass m with control torque u.

The evolution of the pendulum is described by the Euler-Lagrange equation with external force

$$ml^2\ddot{\theta} + mgl\sin\theta = u \ .$$

We rewrite it as

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\tfrac{g}{l}\sin\theta + \tfrac{u}{ml^2}. \end{aligned}$$

Denote $x_1 = \theta$ and $x_2 = \omega$ and consider the evolution of the pendulum on the state space $\mathbb{R}^2$, that is $x = (x_1, x_2)^T \in \mathbb{R}^2$. We get the system Σ

$$\Sigma : \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{l}\sin x_1 + \tfrac{u}{ml^2}. \end{aligned}$$

Replace the control u by

$$u = ml^2\tilde{u} + mlg\sin x_1,$$

which can be interpreted as a transformation in the control space U depending on the state $x \in X$. We get the linear control system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{u}. \end{aligned}$$

Using a simple transformation in the control space we thus brought the system into the simplest possible form: a linear one. Notice that the families of all trajectories of both systems coincide although they are parametrized (with respect to the control parameters u and $\tilde{u}$, respectively) in two different ways.

Now fix an angle θ_0. The goal is to stabilize the system around $x_0 = (x_{10}, x_{20})^T$, where $x_{10} = \theta_0$ and $x_{20} = 0$. Introduce new coordinates

$$\begin{aligned} \tilde{x}_1 &= x_1 - x_{10} \\ \tilde{x}_2 &= x_2. \end{aligned}$$

and apply the control

$$\tilde{u} = k_1\tilde{x}_1 + k_2\tilde{x}_2,$$

where k_1, k_2 are real parameters to be chosen. We get a *closed loop system* described by the system of linear differential equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= k_1\tilde{x}_1 + k_2\tilde{x}_2, \end{aligned}$$

whose characteristic polynomial is given by

$$p(\lambda) = \lambda^2 - \lambda k_2 - k_1.$$

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be any pair of conjugated complex numbers. Take

$$\begin{aligned} k_1 &= -\lambda_1\lambda_2 \\ k_2 &= \lambda_1 + \lambda_2, \end{aligned}$$

then the eigenvalues of the closed loop system are λ_1 and λ_2. In particular, by choosing λ_1 and λ_2 in the left half plane we stabilize exponentially the pendulum around an arbitrary angle θ_0 and a stabilizing control can be taken as

$$u = k_1 ml^2(x_1 - x_{10}) + k_2 ml^2 x_2 + mgl \sin x_1.$$

Now fix for the system Σ an initial point $x_0 = (x_{10}, x_{20})^T \in \mathbb{R}^2$ and a terminal point $x_T = (x_{1T}, x_{2T})^T \in \mathbb{R}^2$ and consider the problem of finding a control $u(t)$, $0 \le t \le T$, which generates a trajectory $x(t)$, $0 \le t \le T$, such that $x(0) = x_0$ and $x(T) = x_T$. This is the *controllability problem*, called also *motion planning problem*. Due to the above described linearization, we get the following simple solution of the problem. Choose an arbitrary C^2-function $\varphi(t)$, $0 \le t \le T$, such that

$$\begin{aligned} \varphi(0) &= x_{10} \\ \varphi'(0) &= x_{20} \\ \varphi(T) &= x_{1T} \\ \varphi'(T) &= x_{2T}. \end{aligned}$$

and apply to the system the control

$$\tilde{u}(t) = \varphi''(t)$$

or, equivalently,

$$u(t) = ml^2\varphi''(t) + mlg\sin x_1(t).$$

Clearly, the proposed control solves the motion planning problem producing a trajectory that joins x_0 and x_T. □

Now consider a single-input linear control system of the form

$$\Lambda: \ \dot{x} = Ax + bu,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and assume that Λ is controllable, that is

$$\text{rank}\ \big(b, Ab, \ldots, A^{n-1}b\big) = n.$$

Choose a linear function $h = cx$, where c is a row vector, such that

$$cb = cAb = \cdots = cA^{n-2}b = 0$$

and

$$cA^{n-1}b = d \neq 0,$$

whose existence follows immediately from the controllability assumption. Introduce linear coordinates

$$\begin{array}{rcl} \tilde{x}_1 & = & cx \\ \tilde{x}_2 & = & cAx \\ & \vdots & \\ \tilde{x}_n & = & = cA^{n-1}x. \end{array}$$

We have

$$\begin{array}{rclclcl} \dot{\tilde{x}}_1 & = & c\dot{x} & = & cAx + cbu & = & \tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & cA\dot{x} & = & cA^2x + cAbu & = & \tilde{x}_3 \\ & \vdots & & & & & \\ \dot{\tilde{x}}_{n-1} & = & cA^{n-2}\dot{x} & = & cA^{n-1}x + cA^{n-2}bu & = & \tilde{x}_n \\ \dot{\tilde{x}}_n & = & cA^{n-1}\dot{x} & = & cA^nx + cA^{n-1}bu & = & \sum_{i=1}^n a_i\tilde{x}_i + du, \end{array}$$

for some $a_i \in \mathbb{R}$, for $1 \leq i \leq n$. By introducing a new control variable

$$\tilde{u} = \sum_{i=1}^{n} a_i\tilde{x}_i + du,$$

which can be viewed at as a state depending transformation in the control space U, we bring any single-input controllable linear system into the n-fold integrator

$$\dot{\tilde{x}}_1 = \tilde{x}_2,\ \dot{\tilde{x}}_2 = \tilde{x}_3,\ \ldots,\ \dot{\tilde{x}}_{n-1} = \tilde{x}_n,\ \dot{\tilde{x}}_n = \tilde{u}.$$

We will consider the problem of whether and when such a transformation is possible in the nonlinear case. Consider a single-input control affine system of the form

$$\Sigma:\ \dot{x} = f(x) + g(x)u,$$

where $x \in X$, an open subset of $\mathbb{R}^n$, and f and g are C^∞-smooth vector fields on X.

Recall that $L_v\varphi$ denotes the derivative of a function φ with respect to a vector field v, that is

$$L_v\varphi(x) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(x)v_i(x).$$

Fix a point $x_0 \in X$ and assume that there exist a C^∞-smooth function φ on X such that (compare the linear case)

$$L_g\varphi = L_gL_f\varphi = \cdots = L_gL_f^{n-2}\varphi = 0$$

and

$$L_gL_f^{n-1}\varphi(x) = d(x),$$

where $d(x)$ is a smooth function such that $d(x_0) \neq 0$. If around the point x_0, the functions φ, $L_f\varphi$, ... ,$L_f^{n-1}\varphi$ are independent (in the sense that their differentials are linearly independent around x_0), then in a neighborhood V of x_0 the map

$$\begin{array}{rcl} \tilde{x}_1 & = & \varphi \\ \tilde{x}_2 & = & L_f\varphi \\ & \vdots & \\ \tilde{x}_n & = & L_f^{n-1}\varphi \end{array}$$

defines a local diffeomorphism, or, in other words, a local coordinate system. In the local coordinates $(\tilde{x}_1, \ldots, \tilde{x}_n)^T$ we have

$$\begin{array}{rclclcl} \dot{\tilde{x}}_1 & = & < d\varphi, \dot{x} > & = & L_f\varphi + uL_g\varphi & = & \tilde{x}_2 \\ \vdots & & & \vdots & & \vdots & \\ \dot{\tilde{x}}_{n-1} & = & < dL_f^{n-2}\varphi, \dot{x} > & = & L_f^{n-1}\varphi + uL_gL_f^{n-2}\varphi & = & \tilde{x}_n \\ \dot{\tilde{x}}_n & = & < dL_f^{n-1}\varphi, \dot{x} > & = & L_f^{n}\varphi + uL_gL_f^{n-1}\varphi & = & L_f^{n}\varphi + ud(x). \end{array}$$

By introducing a new control variable

$$\tilde{u} = L_f^n \varphi + u L_g L_f^{n-1} \varphi,$$

which can be viewed at as a transformation in the control space U, depending nonlinearly on the state x, we bring our single-input nonlinear system into the n-fold integrator

$$\dot{\tilde{x}}_1 = \tilde{x}_2,\ \dot{\tilde{x}}_2 = \tilde{x}_3,\ \ldots,\ \dot{\tilde{x}}_{n-1} = \tilde{x}_n,\ \dot{\tilde{x}}_n = \tilde{u}.$$

The proposed method works under two assumptions. Firstly, we assumed the existence of a function φ such that $L_g\varphi = L_g L_f \varphi = \cdots = L_g L_f^{n-2}\varphi = 0$. Secondly, we assumed that the functions φ, $L_f\varphi, \ldots, L_f^{n-1}\varphi$ are independent in a neighborhood of x_0. The former is a system of $n-1$ first order partial differential equations. In order to see it, let us consider the two first equations $L_g\varphi = 0$ and $L_g L_f \varphi = 0$, which imply that

$$L_f L_g \varphi - L_g L_f \varphi = 0.$$

Although the expression on the left hand side involves a priori partial derivatives of order two, it depends on partial derivatives of φ of order one only and a direct calculation shows that we can represent it as

$$L_f L_g \varphi - L_g L_f \varphi = L_{[f,g]}\varphi,$$

where the vector field $[f,g]$ is given by

$$[f,g](x) = \mathrm{D}g(x) f(x) - \mathrm{D}f(x) g(x),$$

where $\mathrm{D}g(x)$ (resp. $\mathrm{D}f(x)$) stands for the derivative at x, that is, the Jacobi matrix of the map $g : X \longrightarrow \mathbb{R}^n$ (resp. $f : X \longrightarrow \mathbb{R}^n$). We will call $[f,g]$ the *Lie bracket* of the vector fields f and g. We would like to emphasize two important aspects of the nature of Lie bracket. Firstly, it is a vector field, because if we change coordinates then the Lie bracket is multiplied on the left by the Jacobi matrix of the derivative of the coordinate change. This shows its vector, i.e., contravariant, nature. Secondly, a Lie bracket $[f,g]$ acts on a function φ by the formula $L_{[f,g]}\varphi$, that is, acts as a first order differential operator. Notice that, as we have already said, the expression $L_f L_g \varphi - L_g L_f \varphi$ involves, a priori, second order derivatives of φ but all of them are mixed partials that mutually cancel due to Schwarz lemma.

Introduce the notation

$$\mathrm{ad}_f g = [f, g]$$

and, inductively,

$$\mathrm{ad}_f^{j+1} g = [f, \mathrm{ad}_f^j g],$$

for any integer $j \geq 1$. Put $\mathrm{ad}_f^0 g = g$. It can be shown by an induction argument that the existence of a function φ such that $L_g\varphi = L_g L_f \varphi = \cdots = L_g L_f^{n-2}\varphi = 0$ is equivalent to the solvability of the following system of first order partial differential equations

$$\begin{cases} L_g\varphi &= 0 \\ L_{\mathrm{ad}_f g}\varphi &= 0 \\ & \vdots \\ L_{\mathrm{ad}_f^{n-2} g}\varphi &= 0, \end{cases} \tag{2.1}$$

which in coordinates is expressed as

$$\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} (\mathrm{ad}_f^j g)_i = 0, \quad \text{for } 0 \leq j \leq n-2,$$

where $(\mathrm{ad}_f^j g)_i$ denotes the i-th component, in the coordinates $(x_1, \ldots, x_n)^T$, of the vector field $\mathrm{ad}_f^j g$.

It can be shown that the requirement that the differentials $dL_f^j \varphi$, for $0 \leq j \leq n-1$, where φ is a nontrivial solution of the system (2.1), are linearly independent at x_0 is equivalent to the linear independence of $\mathrm{ad}_f^j g$ at x_0, for $0 \leq j \leq n-1$.

We will show that a necessary condition for the above system of first order PDE's to admit a nontrivial solution is that for any $0 \leq i, j \leq n-2$ the Lie bracket $[\mathrm{ad}_f^i g, \mathrm{ad}_f^j g](x)$ belongs to the linear space generated by $\{\mathrm{ad}_f^q g(x),\ 0 \leq q \leq n-2\}$. In view of the linear independence of the $\mathrm{ad}_f^q g$'s, this is equivalent to the existence of smooth functions α_q^{ij} such that

$$[\mathrm{ad}_f^i g, \mathrm{ad}_f^j g] = \sum_{q=0}^{n-2} \alpha_q^{ij} \mathrm{ad}_f^q g.$$

To prove it, assume that there exists a vector field v of the form $v = [\mathrm{ad}_f^i g, \mathrm{ad}_f^j g]$, for some $0 \leq i, j \leq n-2$, and a point $x \in X$, such that $v(x) \notin \mathrm{span}\,\{\mathrm{ad}_f^q g(x),\ 0 \leq q \leq n-2\}$. We have

$$L_v\varphi = L_{[\mathrm{ad}_f^i g, \mathrm{ad}_f^j g]}\varphi = L_{\mathrm{ad}_f^i g} L_{\mathrm{ad}_f^j g}\varphi - L_{\mathrm{ad}_f^j g} L_{\mathrm{ad}_f^i g}\varphi = 0.$$

The n vector fields v and $\mathrm{ad}_f^q g$, for $0 \le q \le n-2$, are linearly independent in a neighborhood of $x \in X$ and therefore the only solutions of the system of n first order PDE's

$$\begin{cases} L_v \varphi &= 0 \\ L_{\mathrm{ad}_f^j g} \varphi &= 0, \quad \text{for } 0 \le j \le n-2, \end{cases}$$

are φ =constant.

It turns out that the two above necessary conditions are also sufficient for the solvability of the problem. Indeed, we have the following result.

Theorem 2.2 *There exist a local change of coordinates $\tilde{x} = \phi(x)$ and a feedback of the form $u = \alpha(x) + \beta(x)\tilde{u}$, where $\beta(x) \ne 0$, transforming, locally around $x_0 \in X$, the nonlinear system*

$$\Sigma: \ \dot{x} = f(x) + g(x)u$$

into a linear controllable system of the form

$$\Lambda: \ \dot{\tilde{x}} = A\tilde{x} + b\tilde{u}$$

if and only if the system Σ satisfies in a neighborhood of x_0:

(C1) $g(x), \mathrm{ad}_f g(x), \dots, \mathrm{ad}_f^{n-1} g(x)$ *are linearly independent;*

(C2) *for any $0 \le i, j \le n-2$, there exist smooth functions α_q^{ij} such that*

$$[\mathrm{ad}_f^i g, \mathrm{ad}_f^j g] = \sum_{q=0}^{n-2} \alpha_q^{ij} \mathrm{ad}_f^q g.$$

The condition (C2), called *involutivity*, is discussed in the general context in the section devoted to Frobenius theorem of [26] in this volume and in the context of feedback linearization in Section 4. It has a clear geometric interpretation. If the above defined system of PDE's $L_g \varphi = \cdots = L_{\mathrm{ad}_f^{n-2} g} \varphi = 0$ admits a nontrivial solution then for any constant $c \in \mathbb{R}$ the equation $\varphi = c$ defines a hypersurface in X. The vectors $g(x), \mathrm{ad}_f g(x), \dots, \mathrm{ad}_f^{n-2} g(x)$ form at any $x \in \{\varphi(x) = c\}$ the tangent space to that hypersurface. In general, such a hypersurface need not exist; the involutivity condition (C2) guarantees its existence.

Especially simple is the planar case, that is, $n = 2$, in which the involutivity follows automatically from the linear independence condition.

Corollary 2.3 *A control-affine planar system*

$$\dot{x} = f(x) + g(x)u,$$

where $x \in \mathbb{R}^2$*, is locally feedback linearizable at* x_0 *if and only if* g *and* $\mathrm{ad}_f g$ *are independent at* x_0.

Example 2.4 (Example 2.1 cont.) We have $f = x_2 \frac{\partial}{\partial x_1}$ and $g = \frac{1}{ml^2}\frac{\partial}{\partial x_2}$. Thus the vector fields g and $\mathrm{ad}_f g = -\frac{1}{ml^2}\frac{\partial}{\partial x_1}$ are independent and hence, by Corollary 2.3, we can conclude feedback linearization of the pendulum, a property which we have established by a direct calculation in Example 2.1. □

3 Equivalence of control systems

The question of feedback linearization discussed in Section 2 is a subproblem of a more general problem of feedback equivalence. In this section we study equivalence of control systems. We start with state space equivalence in Section 3.1 and then we define feedback equivalence in Section 3.2. Various aspects of the problem of feedback linearization will be discussed in Section 4.

3.1 State space equivalence

Two systems are state-space equivalent if they are related by a diffeomorphism (and then also their trajectories, corresponding to the same controls, are related by that diffeomorphism). A question of particular interest is that of when a nonlinear system is equivalent to a linear one. If this is the case the nonlinearities of the considered system are not intrinsic, they appear because of a "wrong" choice of coordinates, and the nonlinear system shares all properties of its linear equivalent.

Consider a smooth nonlinear control system of the form

$$\Sigma \ : \quad \dot{x} = f(x, u),$$

where $x \in X$, an open subset of $\mathbb{R}^n$ (or an n-dimensional manifold) and $u \in U$, an open subset of $\mathbb{R}^m$ (or an m-dimensional manifold). The class of admissible controls $\mathcal{U}$ is fixed and $\mathcal{PC} \subset \mathcal{U} \subset \mathcal{M}$, where $\mathcal{PC}$ denotes the class of piece-wise constant controls with values in U and $\mathcal{M}$ the class of measurable controls with values in U.

Consider another control system of the same form with the same control space U and the same class of admissible controls $\mathcal{U}$

$$\tilde{\Sigma}\ :\quad \dot{\tilde{x}} = \tilde{f}(\tilde{x}, u),$$

where $\tilde{x} \in \tilde{X}$, an open subset of $\mathbb{R}^n$ (or an n-dimensional manifold) and $u \in U$. Analogously to the transformation $\Phi_* g$ of a vector field $g(\cdot)$ by a diffeomorphism Φ, we define the transformation of $f(\cdot, u)$ by Φ. Put

$$(\Phi_* f)(\tilde{p}, u) = \mathrm{D}\Phi(\Phi^{-1}(\tilde{p})) \cdot f(\Phi^{-1}(\tilde{p}), u).$$

We say that control systems Σ and $\tilde{\Sigma}$ are *state space equivalent* (respectively, *locally state space equivalent at points* p *and* $\tilde{p}$) if there exists a diffeomorphism $\Phi : X \to \tilde{X}$ (respectively, a local diffeomorphism $\Phi : X_0 \to \tilde{X}$, $\Phi(p) = \tilde{p}$, where X_0 is a neighborhood of p) such that

$$\Phi_* f = \tilde{f}\ .$$

Put

$$\mathcal{F} = \{f_u \mid u \in U\} \quad \text{and} \quad \tilde{\mathcal{F}} = \{\tilde{f}_u \mid u \in U\}\ ,$$

where $f_u = f(\cdot, u)$ and $\tilde{f}_u = \tilde{f}(\cdot, u)$, that is, $\mathcal{F}$ (resp. $\tilde{\mathcal{F}}$) stands for the family of all vector fields corresponding to constant controls of Σ (resp. of $\tilde{\Sigma}$). (Local) state space equivalence of Σ and $\tilde{\Sigma}$ means simply that

$$\Phi_* f_u = \tilde{f}_u \quad \text{for any } u \in U,$$

i.e., that Φ establishes a correspondence between vector fields defined by constant controls.

Recall the notion of the Lie algebra $\mathcal{L}$ of the system, see the section on controllability and accessibility of [26] in this volume. Assume $\dim \mathcal{L}(p) = \dim \tilde{\mathcal{L}}(\tilde{p}) = n$, which implies that Σ and $\tilde{\Sigma}$ are accessible at p and $\tilde{p}$, respectively.

The following observation shows that (local) state space equivalence is very natural.

Proposition 3.1 *Σ and $\tilde{\Sigma}$ are (locally) state space equivalent if and only if there exists a (local) diffeomorphism Φ which (locally, in neighborhoods of p and $\tilde{p}$) preserves trajectories corresponding to the same controls $u(\cdot) \in \mathcal{U}$, i.e.,*

$$\Phi(\gamma_t^u(p)) = \tilde{\gamma}_t^u(\tilde{p})$$

for any $u(\cdot) \in \mathcal{U}$ *and any* t *for which both sides exist, where* $\gamma_t^u(p)$ *(resp.* $\tilde{\gamma}_t^u(\tilde{p})$*) denotes the trajectory of* Σ *(resp.* $\tilde{\Sigma}$*) corresponding to the control function* $u(\cdot) \in \mathcal{U}$ *and passing by* p *(resp. by* $\tilde{p}$*) for* $t = 0$.

Introduce the following notation for left iterated Lie brackets

$$f_{[u_1 u_2 \ldots u_k]} = [f_{u_1}, [f_{u_2}, \ldots, [f_{u_{k-1}}, f_{u_k}] \ldots]]$$

and analogous for the tilded family. In particular $f_{[u_1]} = f_{u_1}$.

The following result was established by Krener [32] (see also Sussmann [42]).

Theorem 3.2 *Assume that the systems* Σ *and* $\tilde{\Sigma}$ *are analytic and that* $\dim \mathcal{L}(p) = n$ *and* $\dim \tilde{\mathcal{L}}(\tilde{p}) = \tilde{n}$.

(i) Σ *and* $\tilde{\Sigma}$ *are locally equivalent at* p *and* $\tilde{p}$ *if and only if there exists a linear isomorphism of the tangent spaces* $F : T_pX \to T_{\tilde{p}}\tilde{X}$ *such that*

$$F f_{[u_1 u_2 \ldots u_k]}(p) = \tilde{f}_{[u_1 u_2 \ldots u_k]}(\tilde{p}), \tag{3.1}$$

for any $k \geq 1$ *and any* $u_1, \ldots u_k \in U$.

(ii) *Assume, moreover, that* X *and* $\tilde{X}$ *are simply connected and that the Lie algebras* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$ *of* Σ *and* $\tilde{\Sigma}$*, respectively, consist of complete vector fields and satisfy Lie rank condition everywhere. If there exist points* $p \in X$ *and* $\tilde{p} \in \tilde{X}$ *and a linear isomorphism* $F : T_pX \to T_{\tilde{p}}\tilde{X}$ *satisfying* (3.1) *then* Σ *and* $\tilde{\Sigma}$ *are state space equivalent.*

This theorem shows that all information concerning (local) behavior is contained in the values at the initial condition of Lie brackets from $\mathcal{L}$. In a sense (iterative) Lie brackets form invariant (higher order) derivatives of the dynamics of the system and in the analytic case they completely determine its local properties as (higher order) derivatives do for analytic functions.

Consider a control-affine system of the form

$$\Sigma_{\text{aff}} : \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i.$$

Denote $g_0 = f$. Using the above theorem we obtain the following linearization result (compare [38], [42]).

Proposition 3.3 *Consider a control-affine analytic system* Σ_{aff}.

(i) *The system* Σ_{aff} *is locally state space equivalent at* $p \in X$ *to a linear controllable system of the form*

$$\Lambda_{\text{c}}: \quad \dot{x} = Ax + c + Bu = Ax + c + \sum_{i=1}^{m} b_i u_i \ , \quad x \in \mathbb{R}^n \ , \quad u \in \mathbb{R}^m,$$

at $x_0 \in \mathbb{R}^n$ *if and only if*
(E1) $[g_{i_1}, [g_{i_2}, \ldots [g_{i_{k-1}}, g_{i_k}] \ldots]](p) = 0$
for any $k \geq 2$ *and any* $0 \leq i_j \leq m$, $1 \leq j \leq k$, *provided that at least two* i_j*'s are different from zero and*
(E2) $\dim \operatorname{span} \{\operatorname{ad}_f^j g_i(p) \mid 1 \leq i \leq m \ , \ 0 \leq j \leq n-1\}(p) = n.$

(ii) *The system* Σ_{aff} *is locally state space equivalent at* $p \in X$ *to a linear controllable system of the form*

$$\Lambda: \quad \dot{x} = Ax + Bu = Ax + \sum_{i=1}^{m} b_i u_i \ , \quad x \in \mathbb{R}^n \ , \quad u \in \mathbb{R}^m,$$

at $0 \in \mathbb{R}^n$ *if and only if* Σ *satisfies* (E1), (E2) *and* $f(p) = 0$.

(iii) *The system* Σ_{aff} *is globally state space equivalent to a controllable linear system* Λ *on* $\mathbb{R}^n$ *if and only if it satisfies* (E1), (E2), *there exists* $p \in X$ *such that* $f(p) = 0$, *the state space* X *is simply connected and, moreover,*
(E3) *the vector fields* f *and* $g_1, \ldots, g_m$ *are complete.*

Recall that a vector field f is complete if its flow $\gamma_t^f(p)$ is defined for any $(t, p) \in \mathbb{R} \times X$.

3.2 Feedback equivalence

The role of the concept of feedback in control cannot be overestimated and is very well understood, both in the linear and nonlinear cases. We would like to consider it as a way of transforming nonlinear systems in order to achieve desired properties. When considering state-space equivalence the controls remain unchanged. The idea of feedback equivalence is to enlarge state-space transformations by allowing to transform controls as well and to transform them in a way which depends on the state: thus *feeding* the state back to the system.

Consider two general control systems Σ and $\tilde{\Sigma}$ given respectively by $\dot{x} = f(x, u)$, $x \in X$, $u \in U$ and $\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u})$, $\tilde{x} \in \tilde{X}$, $\tilde{u} \in \tilde{U}$. Assume that U and

$\tilde{U}$ are open subsets of $\mathbb{R}^m$. We say that Σ and $\tilde{\Sigma}$ are *feedback equivalent* if there exists a diffeomorphism $\chi\ :\ X \times U \to \tilde{X} \times \tilde{U}$ of the form

$$(\tilde{x}, \tilde{u}) = \chi(x, u) = (\Phi(x), \Psi(x, u))$$

which transforms the first system into the second, i.e.,

$$\mathrm{D}\Phi(x) f(x, u) = \tilde{f}(\Phi(x), \Psi(x, u)).$$

Observe that Φ plays the role of a coordinate change in X and Ψ, called feedback transformation, changes coordinates in the control space in a way which is state dependent.

When studying dynamical control systems with parameters and their bifurcations, the situation is opposite: coordinate changes in the parameters space are state-independent, while coordinate changes in the state space may depend on the parameters.

For the control-affine case, i.e., for systems of the form

$$\Sigma_{\mathrm{aff}}:\quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i = f(x) + g(x) u,$$

where $g = (g_1, \ldots, g_m)$ and $u = (u_1, \ldots, u_m)^T$, in order to preserve the control affine form of the system, we will restrict feedback transformations to control affine ones

$$\tilde{u} = \Psi(x, u) = \tilde{\alpha}(x) + \tilde{\beta}(x) u,$$

where $\tilde{\beta}(x)$ is an invertible $m \times m$ matrix and $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)^T$. Denote the inverse feedback transformation by $u = \alpha(x) + \beta(x)\tilde{u}$. Then feedback equivalence means that

$$\tilde{f} = \Phi_*(f + g\alpha) \quad \text{and} \quad \tilde{g} = \Phi_*(g\beta),$$

where $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m)$.

For control linear systems of the form $\dot{x} = g(x)u = \sum_{i=1}^{m} g_i(x) u_i$, (local) feedback equivalence coincides with (local) equivalence of distributions $\mathcal{G}$ spanned by the vector fields g_i's.

4 Feedback linearization

Since feedback transformations change dynamical behavior of a system they are used to achieve some required properties of the system. In Sections 6.2 and 6.3 we will show how feedback transformations are used to synthesize controls with decoupling properties. In this Section we will study the problem of when a nonlinear system can be transformed to a linear form via feedback. A particular case of feedback linearization of single-input control affine systems has been discussed in Section 2. The interest in feedback linearization is two-fold. Firstly, if one is able to compensate nonlinearities by feedback then the modified system possesses all control properties of its linear equivalent and linear control theory can be used in order to study it and/or to achieve the desired control properties. This shows possible engineering applications of feedback linearization, compare Example 2.1. From mathematical (or system theory) viewpoint, if we would like to classify nonlinear systems under feedback transformations (which define a group action on the space of all systems) then one of the most natural problems is to characterize those nonlinear systems which are feedback equivalent to linear ones. In Section 4.1 we will study feedback linearization of multi-input and general nonlinear systems. In Section 4.2 we will consider linearization using feedback which changes the drift vector field only. Finally, in Section 4.3 we will study the problem of finding the largest possible linearizable subsystem of the given system.

4.1 Static feedback linearization

A general nonlinear control system

$$\Sigma: \ \dot{x} = f(x,u),$$

is *(locally at* (x_0,u_0)*) feedback linearizable* if it is (locally at (x_0,u_0)) feedback equivalent to a controllable linear system Λ_c of the form

$$\Lambda_c: \ \dot{\tilde{x}} = A\tilde{x} + c + B\tilde{u}.$$

Recall the notation

$$\mathcal{F} = \{f_u \ \mid \ u \in U\}.$$

For any $u \in U$, define the following distributions on X which will play the fundamental role in solving the feedback linearization problem.

$$\begin{aligned}
\Delta_1(x,u) &= \operatorname{Im}\frac{\partial f}{\partial u}(x,u) \\
\Delta_2(x,u) &= \Delta_1(x,u) + \operatorname{span}\left[\mathcal{F},\Delta_1\right](x,u) \\
&= \Delta_1(x,u) + \operatorname{span}\left\{\left[f_u,g\right](x,u) \mid f_u\in\mathcal{F},\ g\in\Delta_1\right\}
\end{aligned}$$

and, inductively,

$$\begin{aligned}
\Delta_j(x,u) &= \Delta_{j-1}(x,u) + \operatorname{span}\left[\mathcal{F},\Delta_{j-1}\right](x,u) \\
&= \Delta_{j-1}(x,u) + \operatorname{span}\left\{\left[f_u,g\right](x,u) \mid f_u\in\mathcal{F},\ g\in\Delta_{j-1}\right\}.
\end{aligned}$$

Remark 4.1 For the linear system Λ_c we have

$$\Delta_1 = \operatorname{Im} B \qquad \Delta_j = \operatorname{Im}\left(B,\ldots,A^{j-1}B\right),\ j\geq 0.$$

In the control-affine case, the feedback linearization problem was solved by Jakubczyk and Respondek [27], and independently by Hunt and Su [22] (see Theorem 4.5 below). In the general case we have the follwing result.

Theorem 4.2 *Σ is locally feedback linearizable at (x_0,u_0) if and only if it satisfies in a neighborhood of (x_0,u_0) the following conditions*

(A0) *Δ_1 does not depend on u,*

(A1) dim $\Delta_j(x,u)$ *=const, $j=1,...,n$,*

(A2) *Δ_j are involutive, $j=1,...,n$,*

(A3) $\dim\Delta_n(x_0,u_0)=n$.

Remark 4.3 One can show that if Δ_1 is involutive, of constant rank, and does not depend on u then the successive distributions Δ_j, for $j\geq 2$, do not depend on u either. Thus we can check the involutivity condition (A2) for them for a single value u only (for example for u_0).

In applications, one is often interested in points of equilibria. Denote by Λ a linear system of the form

$$\Lambda\ :\ \dot{\tilde{x}} = A\tilde{x} + B\tilde{u},$$

that is, the system Λ_c with $c=0$.

Corollary 4.4 *Σ_{aff} is locally feedback equivalent at (x_0, u_0) to a controllable linear system Λ at $(0,0)$ if and only if it satisfies the conditions* (A0)-(A3) *and moreover $f(x_0, u_0) \in \Delta_1(x_0, u_0)$.*

Consider feedback equivalence of linear controllable multi-input systems Λ of the form $\dot{x} = Ax + Bu$ (in this case the diffeomorphism $\Phi(x)$ and feedback $\Psi(x,u)$ are taken to be linear with respect to the state and control). As shown by Brunovský [5], complete feedback invariants are the dimensions m_j of Im M^j, where the map $M^j : \mathbb{R}^{mj} \longrightarrow \mathbb{R}^n$ is defined as $[B, AB, \ldots, A^{j-1}B]$. Put $n_0 = 0$ and $n_j = m_j - m_{j-1}$, for $1 \leq j \leq n$. Define

$$\kappa_j = \max\{n_i \mid n_i \geq j\}. \tag{4.1}$$

Observe that $\kappa_1 \geq \cdots \geq \kappa_m$ and $\sum_{i=1}^m \kappa_i = n$. The integers κ_i, called controllability (or Brunovský) indices, form another set of complete invariants of feedback equivalence of linear controllable systems.

Every controllable system Λ with indices $\kappa_1 \geq \cdots \geq \kappa_m$ is feedback equivalent to the system

$$\begin{array}{rcl} \dot{x}_{i,j} & = & x_{i,j+1}\ , \quad \text{for } 1 \leq j \leq \kappa_i - 1, \\ \dot{x}_{i,\kappa_i} & = & u_i\ , \end{array} \tag{4.2}$$

where $1 \leq i \leq m$, called *Brunovský canonical form*, which consists of m independent series of κ_i integrators.

Very often we deal with control-affine systems Σ_{aff}. To state a feedback linearization result for Σ_{aff}, we define the following distributions

$$\begin{array}{rcl} \mathcal{D}^1(x) & = & \text{span}\{g_i(x),\ 1 \leq i \leq m\} \\ \mathcal{D}^j(x) & = & \text{span}\{\text{ad}_f^{q-1} g_i(x),\ 1 \leq q \leq j,\ 1 \leq i \leq m\}, \end{array}$$

for $j \geq 2$. If the dimensions $d_j(x)$ of $\mathcal{D}^j(x)$ are constant (see (A1)' and (B1) below) we denote them by d_j and we define indices ρ_j as follows. Define $d_0 = 0$ and put $r_j = d_j - d_{j-1}$ for $1 \leq j \leq n$. Then (compare (4.1))

$$\rho_j = \max\{r_i \mid r_i \geq j\}. \tag{4.3}$$

If the distributions Δ_j defined earlier in this section are involutive, then they are feedback invariant. If, moreover, the system is affine with respect to controls then, clearly, $\Delta_j = \mathcal{D}^j$, for $j \geq 1$ and, in particular, $\rho_1 \geq \cdots \geq \rho_m$ are feedback invariant. In this case the indices ρ_j coincide with κ_j, the controllability indices of the linear equivalent of Σ.

The following result (see [27] and [22]) describes linearizable control-affine systems.

Theorem 4.5 *The following conditions are equivalent.*

(i) Σ *is locally feedback linearizable at* $x_0 \in \mathbb{R}^n$.

(ii) Σ *satisfies in a neighborhood of* x_0

(A1)' dim $\mathcal{D}^j(x) = const$, for $1 \leq j \leq n$,

(A2)' $\mathcal{D}^j$ *are involutive,* for $1 \leq j \leq n$,

(A3)' dim $\mathcal{D}^n(x_0) = n$.

(iii) Σ *satisfies in a neighborhood of* x_0

(B1) dim $\mathcal{D}^j(x) = const$, for $1 \leq j \leq n$,

(B2) $\mathcal{D}^{\rho_j - 1}$ *are involutive,* for $1 \leq j \leq m$,

(B3) dim $\mathcal{D}^{\rho_1}(x_0) = n$, *where* ρ_1 *is the largest controllability index.*

In the single-input case $m = 1$, the condition (A3) (or, equivalently, (B3)) states that $g(x_0), \ldots, \mathrm{ad}_f^{n-1} g(x_0)$ are independent, which implies that all distributions $\mathcal{D}^j$, for $1 \leq j \leq n$, are of constant rank. In the following Corollary of Theorem 4.5 we thus rediscover Theorem 2.2.

Corollary 4.6 *A scalar input system* Σ *is feedback linearizable if and only if it satisfies*

(C1) $g(x_0), \ldots, \mathrm{ad}_f^{n-1} g(x_0)$ *are independent,*

(C2) $\mathcal{D}^{n-1}$ *is involutive.*

Example 4.7 Consider the following rigid two-link robot manipulator (*double pendulum*); compare, e.g., [6] or [37].

$$\begin{aligned} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= -M^{-1}(x^1)(C(x^1, x^2) + k(x^1)) + M^{-1}(x^1)u \ , \end{aligned}$$

where θ_1 and θ_2 represent the angles (between the horizontal and the first arm and between the arms) and $x^1 = (\theta_1, \theta_2)$, $x^2 = (\dot{\theta}_1, \dot{\theta}_2)$. The control torques applied to the joints are $u = (u_1, u_2)$ and the positive definite symmetric matrix $M(x^1)$ is given by

$$\begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{pmatrix}.$$

The term $k(\theta)$ represents the gravitational force and the term $C(\theta, \dot{\theta})$ reflects the centripetal and Coriolis force.

We have that $\mathcal{D}^1 = \text{span}\{\frac{\partial}{\partial x^2}\} = \text{span}\{\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\}$ is involutive and $\dim \mathcal{D}^2(x^1, x^2) = 4$. Hence the double pendulum is feedback linearizable. A linearizing feedback is given, e.g., by $u = C(x^1, x^2) + k(x^1) + M(x^1)\tilde{u}$. □

Example 4.8 Consider the following model of a permanent magnet stepper motor [46]

$$\begin{aligned}
\dot{x}_1 &= -K_1x_1 + K_2x_3sin(K_5x_4) + u_1 \\
\dot{x}_2 &= -K_1x_2 + K_2x_3cos(K_5x_4) + u_2 \\
\dot{x}_3 &= -K_3x_1sin(K_5x_4) + K_3x_2cos(K_5x_4) - K_4x_3 + K_6sin(4K_5x_4) - \tau_L/J \\
\dot{x}_4 &= x_3\ ,
\end{aligned}$$

where x_1, x_2 denote currents, x_3 denotes the rotor speed and x_4 its position, J is the rotor inertia, and τ_L is the load torque. We see the distributions $\mathcal{D}^1 = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ and $\mathcal{D}^2 = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ are involutive and that $\dim \mathcal{D}^3(x) = 4$ and thus the system is locally (and even globally!) feedback linearizable. □

Example 4.9 The goal of this example is to show how to solve nonlinear problems by transforming the system to an equivalent linear system and solving the linear version of the problem for the linear system. Consider the following system

$$\begin{aligned}
\dot{x} &= y + yz \\
\dot{y} &= z \\
\dot{z} &= u + \sin x,
\end{aligned}$$

where $(x, y, z) \in \mathbb{R}^3$. We want to stabilize it exponentially globally on $\mathbb{R}^3$. Firstly, we show that the system is feedback linearizable. To simplify calculations, replace f by $\tilde{f} = f + \alpha g = f - (\sin x)g = (y + yz, z, 0)^T$. We have $g = (0, 0, 1)^T$, $\text{ad}_{\tilde{f}}g = -(y, 1, 0)^T$, and $[g, \text{ad}_{\tilde{f}}g] = 0$. Thus the distributions $\mathcal{D}^1 = \text{span}\{\frac{\partial}{\partial z}\}$ and $\mathcal{D}^2 = \text{span}\{y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ are involutive. We seek for a function φ whose differential anihilates $\mathcal{D}^2$ which means to find a solution of the following system of 1-st order partial differential equations (compare Section 2)

$$\begin{cases}
\dfrac{\partial\varphi}{\partial z} = 0 \\[2ex]
y\dfrac{\partial\varphi}{\partial x} + \dfrac{\partial\varphi}{\partial y} = 0.
\end{cases}$$

We conclude that φ can be an arbitrary function of $x - \frac{y^2}{2}$ and we choose $\varphi = x - \frac{y^2}{2}$. Therefore we put, see Section 2, $\tilde{x} = x - \frac{y^2}{2}$, $\tilde{y} = L_f\varphi = y$, $\tilde{z} = L_f^2\varphi = z$ and, finally, $u = \tilde{u} - \sin x$. This yields the following linear system

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{y}\\ \dot{\tilde{y}} &= \tilde{z}\\ \dot{\tilde{z}} &= \tilde{u},\end{aligned}$$

which we stabilize on $\mathbb{R}^3$ globally and exponentially via a linear feedback of the form $\tilde{u} = k\tilde{x} + l\tilde{y} + m\tilde{z}$, where the matrix

$$\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ k & l & m\end{pmatrix}$$

is Hurwitz. Therefore the nonlinear feedback

$$u = k(x - \frac{y^2}{2}) + ly + mz - \sin x$$

stabilizes globally and asymptotically on $\mathbb{R}^3$ the original system. □

Example 4.10 Consider the following model of the rigid body whose gas jets control the rotations around the two first principal axes.

$$\begin{aligned}\dot{\omega}_1 &= a_1\omega_2\omega_3 + u_1\\ \dot{\omega}_2 &= a_2\omega_1\omega_3 + u_2\\ \dot{\omega}_3 &= a_3\omega_1\omega_2.\end{aligned}$$

We have $f = (a_1\omega_2\omega_3, a_2\omega_1\omega_3, a_3\omega_1\omega_2)^T$, $g_1 = (1,0,0)^T$ and $g_2 = (0,1,0)^T$. We calculate $\mathrm{ad}_f g_1 = -(0,0,a_3\omega_2)^T$ and $\mathrm{ad}_f g_2 = -(0,0,a_3\omega_1)^T$. We thus see that the distribution $\mathcal{D}^1 = \mathrm{span}\,\{g_1, g_2\} = \mathrm{span}\,\{\frac{\partial}{\partial\omega_1}, \frac{\partial}{\partial\omega_2}\}$ is always involutive and of rank two everywhere while $\mathcal{D}^2 = \mathrm{span}\,\{g_1, g_2, ad_f g_1, ad_f g_2\}$ is of rank three if and only if $a_3 \neq 0$ and either $\omega_1 \neq 0$ or $\omega_2 \neq 0$. In the first case we put $\tilde{\omega}_1 = a_3\omega_1\omega_2$, $u_1 = \frac{1}{a_3\omega_2}(-a_1a_3\omega_2^2\omega_3 - \alpha_2\alpha_3\omega_1^2\omega_3 + a_3\omega_1\tilde{u}_2)$, and $u_2 = -a_2\omega_1\omega_3 + \tilde{u}_2$ and we get the linear system

$$\begin{aligned}\dot{\tilde{\omega}}_1 &= \tilde{u}_1\\ \dot{\omega}_2 &= \tilde{u}_2\\ \dot{\omega}_3 &= \tilde{\omega}_1.\end{aligned}$$

In the second case we we put $\tilde{\omega}_2 = a_3\omega_1\omega_2$ and define $\tilde{u}_1$ and $\tilde{u}_2$ in an anologous way. □

Example 4.11 Consider the following model of unicycle

$$\begin{aligned}\dot{x}_1 &= u_1 \cos\theta \\ \dot{x}_2 &= u_2 \sin\theta \\ \dot{\theta} &= u_2 \ ,\end{aligned}$$

where $(x_1, x_2, \theta) \in \mathbb{R}^2 \times S^1$. We have

$$g_1 = (\cos\theta, \sin\theta, 0)^T, \ g_2 = (0, 0, 1)^T,$$

thus $[g_1, g_2] = (\sin\theta, -\cos\theta, 0)^T$ and hence $\mathcal{D}^1$ is not involutive: the unicycle is *not* static feedback linearizable. □

4.2 Restricted feedback linearization

Consider a control-affine system Σ_{aff} and a feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$ which can be interpreted as an (affine) change of coordinates, depending on the state, in the input space. The term β allows to choose generators of the distribution $\mathcal{D}^1 = \text{span}\{g_1, \dots, g_m\}$ whereas the term α changes the drift f. Restricted feedback allows to transform the drift f only and keeps the g_i's unchanged. More precisely, two control affine systems Σ_{aff} and $\tilde{\Sigma}_{\text{aff}}$ are *restricted feedback equivalent* if there exist a diffeomorphism Φ between their state spaces and a restricted feedback of the form $u = \alpha(x) + \tilde{u}$ such that

$$\tilde{f} = \Phi_*(f + g\alpha) \quad \text{and} \quad \tilde{g}_i = \Phi_* g_i, \tag{4.4}$$

for $1 \leq i \leq m$.

We will be interested in equivalence to liner systems under such feedback and we will call it *restricted feedback linearization.*

The three main reasons to discuss restricted feedback linearization are as follows. Firstly, it was Brockett's restricted feedback linearization result [3] which begun an increasing interest in various kinds of feedback linearization problems for nonlinear systems. Secondly, there is a nice stochastic interpretation of the restricted feedback linearization [3]. Thirdly, it is relatively easy, as we will show it, to proceed from local results to global ones.

Consider single-input systems of the form

$$\Sigma_{\text{aff}} \ : \ \ \dot{x} = f(x) + g(x)u \ , \ x \in X \ , \ \ u \in \mathbb{R},$$

and study their equivalence to linear single-input systems of the form

$$\Lambda_c \ : \quad \dot{\tilde{x}} = A\tilde{x} + c + b\tilde{u} \ , \quad \tilde{x} \in \mathbb{R}^n \quad \tilde{u} \in \mathbb{R}.$$

We have the following result [1].

Theorem 4.12 Σ_{aff} *is locally restricted feedback linearizable at* x_0 *if and only if it satisfies in a neighborhood of* x_0 *the following conditions*

(RC1) $g(x_0), \ldots, \mathrm{ad}_f^{n-1} g(x_0)$ *are independent.*

(RC1) $[\mathrm{ad}_f^q g, \mathrm{ad}_f^r g] \subset \mathcal{D}^{n-2}$ *for any* $0 \leq q, r \leq n-1$,

Remark 4.13 Like in the case of feedback linearization (compare Corollary 4.4), Σ_{aff} is restricted feedback equivalent at x_0 to Λ_c, with $c = 0$, at 0 if and only if $f(x_0) \in \mathcal{D}^1(x_0)$.

In the single-input case all linearizable systems are equivalent to the Brunovský canonical form (compare (4.2))

$$\begin{aligned} \dot{x}_i &= x_{i+1} \ , \ \text{for } 1 \leq i \leq n-1 \ , \\ \dot{x}_n &= u \ . \end{aligned} \tag{4.5}$$

If Σ_{aff} is (locally) feedback linearizable, then there are many pairs (α, β) and many (local) diffeomorphisms which transform Σ_{aff} into its Brunovský canonical form. However, if we allow for restricted feedback only, then α transforming Σ_{aff} into the canonical form is unique and is given by

$$\alpha = (-1)^{n-1} L_f^{n-1} \gamma_n \ , \tag{4.6}$$

where L_f stands for the Lie derivative along f and the smooth function γ_n is uniquely defined by

$$f = \sum_{i=1}^{n} \gamma_i \mathrm{ad}_f^{i-1} g.$$

This observation is crucial for establishing the following result on restricted feedback linearization [9], [39].

Theorem 4.14 Σ *is restricted feedback globally linearizable, that is, globally equivalent via a restricted feedback to a linear system on* $\mathbb{R}^n$*, if and only if it satisfies the conditions* (RC1),(RC2) *and, moreover,*

(RC3) *the vector fields* $\tilde{f}$ *and* g *are complete, where* $\tilde{f} = f + g\alpha$ *and* α *is defined by* (4.6),

(RC4) *the state space* X *is simply connected.*

Example 4.15 (Continuation of Examples 2.1 and 2.4). We have $f = \omega\frac{\partial}{\partial\theta} - \frac{g}{l}\sin\theta\frac{\partial}{\partial\omega}$ and $g = \frac{1}{ml^2}\frac{\partial}{\partial\omega}$. Therefore $[\mathrm{ad}_f g, g] = 0$ and since g and $\mathrm{ad}_f g$ are independent everywhere, the system is restricted feedback linearizable. Indeed, it is immediate to see that the feedback $u = mgl\sin\theta + \tilde{u}$ brings the system to a linear form (no action of diffeomorphism is needed). □

The nonlinear pendulum defined on $S^1 \times \mathbb{R}^1$ is globally equivalent to a linear system evolving on $S^1 \times \mathbb{R}^1$. If we enlarge the class of linear systems to include systems of the form $\dot{x} = Ax + Bu$, where each component x_i of x is either a global coordinate on $\mathbb{R}^1$ or a global coordinate (angle) on S^1, then Theorem 4.14 remains true if we drop the assumption (RC4). This includes many mechanical control systems.

4.3 Partial linearization

The linearizability conditions are restrictive (except for the scalar input affine systems on the plane, compare Corollary 2.3). Given a nonlinearizable system it is therefore natural to ask what is its largest linearizable subsystem. Consider a partially linear system Λ_{part} of the form

$$\Lambda_{\text{part}}: \quad \begin{array}{rcl} \dot{\tilde{x}}^1 & = & A\tilde{x}^1 + c + \sum_{i=1}^m b_i \tilde{u}_i \\ \dot{\tilde{x}}^2 & = & \tilde{f}^2(\tilde{x}^1, \tilde{x}^2) + \sum_{i=1}^m \tilde{g}_i^2(\tilde{x}^1, \tilde{x}^2)\tilde{u}_i, \end{array}$$

with $\tilde{x}^1$, $\tilde{x}^2$ being possibly vectors. Recall the notion of the Lie ideal $\mathcal{L}_0$ of the system (see [26] of this volume), which is defined as the Lie ideal generated by $g_1, \ldots, g_m$ in $\mathcal{L}$, or, in other words, $\mathcal{L}_0 = \mathrm{Lie}\,\{\mathrm{ad}_f^q g_i \mid 1 \le i \le m,\ q \ge 0\}$. With the help of $\mathcal{L}_0$ we define another Lie ideal by putting

$$\mathcal{L}^2 = [\mathcal{L}_0, \mathcal{L}_0] = \{[f_1, f_2] \ \mid \ f_1, f_2 \in \mathcal{L}_0\}\,.$$

It is $\mathcal{L}^2$ which contains all intrinsic nonlinearities not removable by the action of diffeomorphisms as the following result [40] shows.

Theorem 4.16 *Consider a control affine system* Σ_{aff}*.*

(i) *If* Σ_{aff} *is locally state space equivalent at* x_0 *to a partially linear system* Λ_{part} *then* $\dim \mathcal{L}^2(x) < n$ *in a neighborhood of* x_0*.*

(ii) *Assume that* Σ_{aff} *satisfies* $\dim \mathcal{L}_0(x_0) = n$ *and that* $\dim \mathcal{L}^2(x) = \sigma$*=const. in a neighborhood of* x_0*. Then* Σ_{aff} *is locally state space equivalent to a partially linear system* Λ_{part}*, such that the dimension* *of the linear subsystem is* $\dim \tilde{x}^1 = n - \sigma$ *and, moreover, the linear subsystem is controllable.*

Corollary 4.17 *Let an analytic system* Σ_{aff} *satisfies* $\dim \mathcal{L}_0(x_0) = n$. *It is locally state space equivalent at* x_0 *to a partially linear system* Λ_{part} *if and only if*

$$\dim \mathcal{L}^2(x_0) < n.$$

Moreover, there exists a system Λ_{part}, *with* $n-\sigma$*-dimensional linear controllable subsystem, where* $\sigma = \dim \mathcal{L}^2(x_0)$, *which is state space equivalent to* Σ_{aff}.

Now we consider the problem of transforming a nonlinear system to a partially linear one via feedback. This problem has been studied and solved in the scalar-input case in [33] and in the multi-input case in [34] and [40]. Recall that for a smooth distribution $\mathcal{D}$ we denote by $\overline{\mathcal{D}}$ its involutive closure, that is, the smallest distribution containing $\mathcal{D}$ and closed under the Lie bracket.

Theorem 4.18 *Consider a single-input system* Σ_{aff}.

(i) *If* Σ_{aff} *is locally feedback equivalent at* x_0 *to a partially linear* Λ_{part} *with* ρ*-dimensional linear controllable subsystem then* Σ_{aff} *satisfies the following conditions:*

(PC1) $g(x_0), \dots, \text{ad}_f^{\rho-1}(x_0)$ *are independent,*

(PC2) $\dim \overline{\mathcal{D}}^{\rho-1}(x) < n$ *in a neighborhood of* x_0,

(PC3) $\text{ad}_f^{\rho-1} g(x) \notin \overline{\mathcal{D}}^{\rho-1}(x)$.

(ii) *Assume that there exists an integer* ρ *such that* $\dim \overline{\mathcal{D}}^{\rho-1}(x) = const.$ *and that* (PC1), (PC2), (PC3) *are satisfied. Then* Σ_{aff} *is locally feedback equivalent to a partially linear system* Λ_{part} *with* ρ*-dimensional linear controllable subsystem, that is* $\dim \tilde{x}^1 = \rho$. *Moreover, the largest* ρ *satisfying the above conditions gives the largest dimension of linear subsystem among all possible partial linearizations.*

Example 4.19 Consider a symmetric rigid body (two inertia momenta are equal) with one pair of jets

$$\begin{array}{rcl} \dot{\omega}_1 &=& a\omega_2\omega_3 + e_1 u \\ \dot{\omega}_2 &=& -a\omega_1\omega_3 + e_2 u \\ \dot{\omega}_3 &=& e_3 u \quad . \end{array}$$

Compute $g = (e_1, e_2, e_3)^T$, $\mathrm{ad}_f g = a(e_2\omega_3 + e_3\omega_2, -e_1\omega_3 - e_3\omega_1, 0)^T$. Hence for $\mathcal{D}^2$ to be involutive, that is, $[g, \mathrm{ad}_f g] = 2ae_3(-e_2, e_1, 0) \in \mathcal{D}^2 = \mathrm{span}\{g, \mathrm{ad}_f g\}$, we need either $e_3 = 0$ or $e_1 = e_2 = 0$. In the former case, ω_3 remains constant, in the latter, the symmetric spacecraft is controlled in a symmetric way: the angular momentum of the jet is parallel to the third principal axis. Notice that for all values of the control vector $e = (e_1, e_2, e_3)^T$, the system is not feedback linearizable. Indeed, either $\mathcal{D}^2$ is not involutive or the system is not accessible. On the other hand, for all values of the control vector field $e \neq 0$, the system contains a 2-dimensional linear subsystem for an open and dense set of initial conditions. □

5 Observability

In this chapter we consider briefly the concept of nonlinear observability. We start with geometric approach to the observability problem and in Section 5.1 we state a sufficient condition, called observability rank condition, based on successive Lie derivatives of the output along the dynamics. In Section 5.2 we discuss (local) decompositions into observable and completely unobservable parts which generalize the classical Kalman decomposition. Then in Section 5.3 we consider the problem of uniform observability, which means that we can observe the system for any input. In Section 5.4 we give a necessary and sufficient condition for local observability. Finally, in Section 5.5 we discuss generic properties: we give normal forms for generic systems and recall results concerning genericity of observability.

5.1 Nonlinear observability

Consider the class of nonlinear systems with outputs (measurements) of the form

$$\Sigma : \quad \begin{array}{rcl} \dot{x} &=& f(x,u)\,, \\ y &=& h(x)\,, \end{array}$$

where $x \in X$, $u \in U$, $y \in Y$. Here X, U, and Y are open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$, respectively (or differentiable manifolds of dimensions n, m, and p, respectively)[1]. The map $h : X \to Y$ represents the vector of p measurements (observations), where $h_i \in C^\infty(X)$, for $1 \leq i \leq p$, and $h = (h_1, \dots, h_p)^T$.

[1]Except for the second part of Section 5.5, where we assume U to be J^m, with J being a compact subinterval of $\mathbb{R}$.

Throughout this section, Σ will denote the above described nonlinear system with output.

The class of admissible controls $\mathcal{U}$ is fixed and $\mathcal{PC} \subset \mathcal{U} \subset \mathcal{M}$, where $\mathcal{PC}$ denotes the class of piece-wise constant controls with values in U and $\mathcal{M}$ the class of measurable controls with values in U.

Let $\mathcal{Y}$ denote the space of absolutely continuous functions on X with values in Y. For the system Σ we define the *response map*, called also *input-output map*,

$$R_\Sigma : X \times \mathcal{U} \longrightarrow \mathcal{Y},$$

which to any initial condition $q \in X$ and any admissible control $u(\cdot) \in \mathcal{U}$ attaches the output of the system

$$y_{q,u}(t) = y(t, q, u(\cdot)) = h(x(t, q, u(\cdot))),$$

where $x(t, q, u(\cdot))$ denotes the solution of $\dot{x} = f(x, u)$, for $u(\cdot) \in \mathcal{U}$, passing through q, that is $x(0, q, u(\cdot)) = q$. The control $u(\cdot)$ being defined on an interval $I_u \subset \mathbb{R}$, such that $0 \in I_u$, we consider the output $y(\cdot)$ on the maximal interval $I_y \subset I_u \subset \mathbb{R}$ on which it exists.

Roughly speaking, the problem of observability is that of the injectivity, with respect to the initial condition, of the response map.

We say that two states $q_1, q_2 \in X$ are *indistinguishable*, and we write $q_1 I q_2$, if

$$y_{q_1,u}(t) = y_{q_2,u}(t),$$

for any $u(\cdot) \in \mathcal{U}$ and any t for which both sides exist.

Definition 5.1 We call the system Σ *observable* if for any two states $q_1, q_2 \in X$ we have

$$q_1 I q_2 \Longrightarrow q_1 = q_2,$$

that is, if there exists an admissible control $u(\cdot) \in \mathcal{U}$ and a time $t \geq 0$ such that

$$y_{q_1,u}(t) \neq y_{q_2,u}(t).$$

meaning that the states q_1 and q_2 are *distinguishable.*

Definition 5.2 Σ is called *locally observable* at $q \in X$ if there is a neighborhood V of q such that for any $\tilde{q} \in V$, the states q and $\tilde{q}$ are distinguishable.

Given a system Σ and an open set $V \subset X$, by the restriction $\Sigma_{|V}$ we will mean a control system with the state space V, defined by the restrictions of f and h to $V \times U$ and V, respectively.

Definition 5.3 Σ is called *strongly locally observable* at a point $q \in X$ if there exists a neighborhood V of q such that the restricted system $\Sigma_{|V}$ is observable.

We would like to emphasize some features of the introduced concepts of observability. Strong local observability is a local concept in two aspects. Firstly, strong local observability means that, in general, we are able to distinguish neighboring points only. Secondly, we are able to do so considering trajectories which stay close to the initial condition. Of course, observability implies strong local observability at any point (we can take $V = X$), which, in turn, implies local observability at any point (for each point we take the neighborhood V existing due to the strong local observability). In general, the reversed implications do not hold, see Examples 5.6 and 5.7 below.

Example 5.4 Consider a mechanical system evolving according to Newton's law

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u,$$

where x_1 denotes the position, x_2 the velocity, and u is the control force. We observe the position

$$y = x_1.$$

This system is clearly observable. Indeed, let $y_{q,u}(t)$ and $\tilde{y}_{\tilde{q},u}(t)$ be the outputs of the system initialized, respectively, at $q = (x_{10}, x_{20})^T$ and at $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20})^T$ and governed by a control $u(\cdot)$. Assume that $y_{q,u}(t) = \tilde{y}_{\tilde{q},u}(t)$ for any t. Then comparing at $t = 0$ both sides of the above equality as well as derivatives at $t = 0$ of both sides, we get $x_{10} = \tilde{x}_{10}$ and $x_{20} = \tilde{x}_{20}$, which proves the observability.

Now assume that, for the same control system, we observe the velocity

$$y = x_2.$$

The system is not observable. Indeed, the initial conditions $q = (x_{10}, x_{20})^T$ and $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20})^T$, such that $x_{10} \neq \tilde{x}_{10}$ but $x_{20} = \tilde{x}_{20}$, produce the same output $y(t) = \int_0^t u(s)ds + x_{20}$. Mechanically, this is obvious: we cannot estimate the position if we observe the velocity only. □

Example 5.5 Consider the linear oscillator (linear pendulum) given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1,\end{aligned}$$

where x_1 denotes the position and x_2 the velocity. Assume that we observe the position

$$y = x_1.$$

Then the system is observable and, given the output function $y(\cdot)$, we can deduce the initial condition $(x_{10}, x_{20})^T$ by taking, like in Example 5.4, the output and its first derivative with respect to time at $t = 0$.

Now assume that we observe the velocity

$$y = x_2.$$

This system is also observable and once again we can deduce the initial condition by looking at the values at $t = 0$ of the output and its first time derivative. The reason for which observing the velocity renders the system observable is that the evolution of the velocity x_2 depends on the position x_1, which is not the case of the system of Example 5.4. □

Example 5.6 Consider the unicycle

$$\begin{aligned}\dot{x}_1 &= u_1 \cos\theta, & y_1 &= x_1\\ \dot{x}_2 &= u_1 \sin\theta, & y_2 &= x_2\\ \dot{\theta} &= u_2,\end{aligned}$$

where $(x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}$ is the position of the center of the mass of the unicycle and $\theta \in S^1$ is the angle between the horizontal and the axis of the unicycle. We observe the position of the center of the mass.

The unicycle is observable. To see it, consider the outputs $y_{q,u}(t)$ and $\tilde{y}_{\tilde{q},u}(t)$ of the system controlled by $u(t) = (u_1(t), u_2(t))^T$, such that $u_1(t) \equiv 1$, passing for $t = 0$ by $q = (x_{10}, x_{20}, \theta_0)^T$ and $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20}, \tilde{\theta}_0)^T$, respectively. Assume that $y_{q,u}(t) = \tilde{y}_{\tilde{q},u}(t)$. Thus $x_{10} = \tilde{x}_{10}$, $x_{20} = \tilde{x}_{20}$, and moreover $\sin\theta(t) = \sin\tilde{\theta}(t)$ and $\cos\theta(t) = \cos\tilde{\theta}(t)$. Hence we conclude that $\theta_0 = \tilde{\theta}_0$, where θ_0, $\tilde{\theta}_0 \in S^1$.

Now consider the unicycle, with the same observations $y_1 = x_1$ and $y_2 = x_2$, evolving on $\mathbb{R}^3$, that is, we consider $\theta \in \mathbb{R}$. It turns out that the system is not observable. To see this, we will show that the outputs $y_{q,u}(t)$ and $\tilde{y}_{\tilde{q},u}(t)$ of the system coincide for $q = (x_{10}, x_{20}, \theta_0)^T$ and $\tilde{q} =$

$(\tilde{x}_{10}, \tilde{x}_{20}, \tilde{\theta}_0)^T$ such that $x_{10} = \tilde{x}_{10}$, $x_{20} = \tilde{x}_{20}$, and $\tilde{\theta}_0 = \theta_0 + 2k\pi$. We have $\theta(t) = \int_0^t u_2(s)ds + \theta_0$ and $\tilde{\theta}(t) = \int_0^t u_2(s)ds + \tilde{\theta}_0$ and hence $\tilde{\theta}(t) = \theta(t) + 2k\pi$. Thus $\sin\tilde{\theta}(t) = \sin\theta(t)$ and $\cos\tilde{\theta}(t) = \cos\theta(t)$ implying that $y_{q,u}(t) = \tilde{y}_{\tilde{q},u}(t)$, for any control $u(\cdot) \in \mathcal{U}$ and the initial conditions as above. In other words, the points $q = (x_{10}, x_{20}, \theta_0)^T$ and $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20}, \tilde{\theta}_0)^T$ such that $x_{10} = \tilde{x}_{10}$, $x_{20} = \tilde{x}_{20}$, and $\tilde{\theta}_0 = \theta_0 + 2k\pi$ are indistinguishable. Of course, the system is strongly locally observable at any $q \in \mathbb{R}^3$. □

Example 5.7 Consider the system

$$\dot{x} = 0\ , \quad y = x^2\ ,$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}$. The system is not observable because the initial conditions x_0 and $-x_0$ give the same output trajectories. This system is strongly locally observable at any $x_0 \neq 0$. Notice that it is locally observable at any point, in particular at $0 \in \mathbb{R}^2$, although in any neighborhood of 0 there are indistinguishable states. This shows that local observability is indeed a weaker property than strong local observability. □

We will give now a sufficient condition for strong local observability. To this end, we will introduce the following concepts.

Definition 5.8 The *observation space* of Σ is defined as

$$H = \mathrm{span}_{\mathbb{R}}\, \{L_{f_{u_k}} \cdots L_{f_{u_1}} h_i \ \mid \ 1 \leq i \leq p,\ k \geq 0,\ u_1, \ldots, u_k \in U\},$$

where $f_{u_j}(\cdot) = f(\cdot, u_j)$ and $L_g\varphi$ stands for the Lie derivative of a smooth function φ with respect to a smooth vector field g, i.e.,

$$L_g\varphi(x) = d\varphi(x) \cdot g(x).$$

Observe that H is the smallest linear subspace of $C^\infty(X)$ containing the observations $h_1, \ldots, h_p$ and closed with respect to Lie differentiation by all elements of $\mathcal{F} = \{f(\cdot, u), u \in U\}$, i.e., all vector fields corresponding to constant controls. Using functions from H we define the following codistribution

$$\mathcal{H} = \mathrm{span}\ \{d\phi\ :\ \phi \in H\}\,.$$

Notice that, in general, $\mathcal{H}$ is not of constant rank.

In the case of control affine systems of the form

$$\Sigma_{\text{aff}} \ : \quad \begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^{m} g_i(x) u_i \,, \\ y & = & h(x) \,, \end{array}$$

we have

$$\mathcal{H} = \text{span} \left\{ dL_{g_{j_k}} \cdots L_{g_{j_1}} h_i \ : \ 1 \le i \le p, \ 0 \le j_l \le m \right\},$$

where $g_0 = f$.

The following result of Hermann and Krener [21] gives a fundamental criterion for nonlinear observability.

Theorem 5.9 *Assume that the system Σ satisfies*

$$\dim \mathcal{H}(q) = n. \tag{5.1}$$

Then Σ is strongly locally observable at q.

The condition (5.1) will be called *observability rank condition*. It can be considered as a counterpart of the accessibility and strong accessibility rank conditions (see, e.g., the survey [26] of this volume), although the duality is not perfect, as we will see in the next example.

Example 5.10 The converse of Theorem 5.9 does not hold (even in the analytic case) as the following simple example shows. Consider

$$\dot{x} = 0 \,, \quad y = x^3 \,,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$. Of course, the system is strongly locally observable at any $x \in \mathbb{R}$ (even observable on $\mathbb{R}$) but it does not satisfy the rank condition at $0 \in \mathbb{R}$, since we have $\mathcal{H} = \text{span}\{ x^2 dx \}$. This shows also that the rank of $\mathcal{H}$ need not be constant. This is to be compared with the accessibility rank condition, which, in the analytic case, is necessary and sufficient for accessibility. □

Example 5.11 Consider a linear control system with outputs of the form

$$\Lambda \ : \quad \begin{array}{rcl} \dot{x} & = & Ax + Bu \,, \\ y & = & Cx \,, \end{array}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. We have $f = Ax$, $g_k = b_k$, for $1 \leq k \leq m$, and $h_i = C_i x$, for $1 \leq i \leq p$, where C_i denotes the i-th row of the matrix C. We calculate

$$L_f^j h_i = C_i A^j x \quad \text{and} \quad L_{g_k} L_f^j h_i = C_i A^j b_k.$$

Thus $\mathcal{H}(q) = \operatorname{span}\{C_i A^j \mid 1 \leq i \leq p,\ 0 \leq j \leq n-1\}$ and $\dim \mathcal{H}(q) = \operatorname{rank} O$, where O is the Kalman observability matrix

$$O = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix}.$$

Therefore a linear system satisfies the observability rank condition if and only if it satisfies Kalman observability condition $\operatorname{rank} O = n$. In this case, as it follows from Theorem 5.9, the system is strongly locally observable. Moreover, we know from the linear control theory, see e.g. [30], that the system is observable. Indeed, the response map R_Λ of the linear system Λ given by

$$y(t) = Cx(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds$$

and associating to an initial condition x_0 the output trajectory, is affine with respect to the initial condition x_0 and thus local injectivity implies global injectivity. Notice that observability properties of a linear system do not depend on the chosen control; indeed, they depend only on the injectivity of the map

$$x_0 \longmapsto Ce^{At}x_0.$$

In the next example we will show that this no longer true in the nonlinear case. □

Example 5.12 The aim of this example is to show that, contrary to the linear case, controls play an important role in the nonlinear observability. In general, there may exist controls which do not distinguish points nevertheless the system can be observable if other controls distinguish. To illustrate that phenomenon, consider the bilinear system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_2 u\ , \quad y = x_1 \\ \dot{x}_2 &= 0\ , \end{aligned}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$. This system is observable, because if we put $u(t) \equiv 0$ we get an observable linear system. Notice, however, that the constant control $u(t) \equiv 1$ does not distinguish x_0 and $\tilde{x}_0$ such that $x_{10} = \tilde{x}_{10}$ and $x_{20} \neq \tilde{x}_{20}$ (we will come back to this phenomenon in Section 5.3). Of course, we can deduce strong local observability at any point from the rank condition. Indeed, we have $h = x_1$, $f = x_2 \frac{\partial}{\partial x_1}$, and $L_f h = x_2$. Hence $\mathcal{H} = \operatorname{span}\{dx_1, dx_2\}$. □

Example 5.13 Consider the unicycle, see Example, 5.6, for which we observe $y_1 = x_1$ and $y_2 = x_2$. We have $h_1 = x_1$, $h_2 = x_2$, $g_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2}$, and $g_2 = \frac{\partial}{\partial \theta}$. Hence $L_{g_1} h_1 = \cos\theta$ and $L_{g_1} h_2 = \sin\theta$. Thus $\mathcal{H} = \operatorname{span}\{dx_1, dx_2, d\sin\theta, d\cos\theta\}$ implying that $\dim \mathcal{H}(q) = 3$, for any $q \in \mathbb{R}^2 \times S^1$. Therefore the unicycle satisfies the observability rank condition at any point of its configuration space. □

5.2 Local decompositions

Let us start with linear systems of the form

$$\Lambda : \quad \begin{aligned} \dot{x} &= Ax + Bu\,, \\ y &= Cx\,, \end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. Denote by W, the kernel of the linear map defined by the Kalman observability matrix O (see Example 5.11). If Λ is not observable then we can find new coordinates (x^1, x^2), with x^1 and x^2 being possibly vectors and $\dim x^1 = k$, where $\dim W = n - k$, such that $x \in W$ if and only if $x = (0, x^2)$. Then Λ reads

$$\begin{aligned} \dot{x}^1 &= A^1 x^1 + \phantom{A^{22}x^2} + B^1 u\,, \quad y = C^1 x^1\,, \\ \dot{x}^2 &= A^{21} x^1 + A^{22} x^2 + B^2 u\,, \end{aligned}$$

where the pair (C^1, A^1) is observable.

As a consequence, any two initial states whose difference is not in W are distinguishable from each other, in particular, by means of the output produced by the zero input. Contrary, if their difference is in W, then they are indistinguishable. The factor system $\Lambda_{/I}$, where I is the indistinguishability equivalence relation, is observable and is given by

$$\Lambda^1 \ : \ \dot{x}^1 = A^1 x^1 + B^1 u\,,\ y = C^1 x^1.$$

Geometrically, Λ^1 is obtained by factoring the system through the subspace W and the factor system is well defined since W is invariant under A. A natural question is whether we can proceed similarly for the nonlinear system Σ?

Theorem 5.14 *Consider the nonlinear system Σ. Assume that the distribution $\mathcal{H}$ is of constant rank equal to k locally around q. Then we have.*

(i) *The codistribution $\mathcal{H}$ is integrable and there exist local coordinates $(x^1, x^2)^T$ defined in a neighborhood V of q, with x^1, x^2 possibly being vectors, such that $\mathcal{H} = \operatorname{span}\{dx_1^1, \ldots, x_k^1\}$.*

(ii) *In the local coordinates $(x^1, x^2)^T$, the system Σ takes the form*

$$\begin{aligned} \dot{x}^1 &= f^1(x^1, u) , \qquad y = h^1(x^1) , \\ \dot{x}^2 &= f^2(x^1, x^2, u). \end{aligned}$$

(iii) *By taking V sufficiently small, two pints $q, \tilde{q} \in V$ are indistinguishable for $\Sigma_{|V}$ if and only if $\tilde{q} \in S_q$, where S_q is the integral leaf, passing through q, of the codistribution $\mathcal{H}$ restricted to V.*

(iv) *In V, factoring the system through the foliation of the integrable codistribution $\mathcal{H}$, produces the strongly locally observable system Σ^1 which, in $(x^1, x^2)^T$- coordinates, is given by*

$$\Sigma^1 \ : \ \dot{x}^1 = f^1(x^1, u) , \qquad y = h^1(x^1).$$

This result says that locally and under the constant rank assumption, the leaves of the foliation of the integrable codistribution $\mathcal{H}$ consist of indistinguishable points and that, on the other hand, we can distinguish the leaves.

From Theorem 5.14 we immediately get the two following corollaries.

Corollary 5.15 *If $\mathcal{H}$ is of constant rank in a neighborhood of q then the following conditions are equivalent.*

(i) *Σ is locally observable at q.*

(ii) *Σ is strongly locally observable at q.*

(iii) $\dim \mathcal{H}(q) = n$.

Corollary 5.16 *If Σ is locally observable at any point of X then* $\dim \mathcal{H}(q) = n$*, for $q \in X'$, an open and dense subset of X.*

An important case when the observability rank is constant is given by the following.

Proposition 5.17 *Assume that an analytic control system Σ satisfies the accessibility Lie rank condition everywhere on X. Then $\mathcal{H}$ is of constant rank on X. In particular, the system is locally observable at q (or, equivalently, strongly locally observable at q) if and only if* $\dim \mathcal{H}(q) = n$.

To illustrate the decomposition result of this section we consider the following example.

Example 5.18 Consider the unicycle, see Example 5.6, for which we measure the angle θ only, that is $h = \theta$. We have $L_{g_1}h = L_{g_2}h = 0$. Thus $\mathcal{H} = \text{span}\,\{d\theta\}$ defines the foliation

$$\{\theta = \text{const.}\}\,,$$

whose leaves consist of indistinguishable points. Indeed, if $\theta_0 = \tilde{\theta}_0$ then the points $q = (x_{10}, x_{20}, \theta_0)^T$ and $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20}, \tilde{\theta}_0)^T$ are indistinguishable. The obvious reason for this is that the evolution of the observed variable $y(t) = \theta(t)$ is independent of that of $x_1(t)$ and $x_2(t)$. □

5.3 Uniform observability

In Example 5.12 we pointed out that for observable nonlinear systems there may exist controls that render the system unobservable. In this section we describe a class of systems, for which all controls distinguish points.

Definition 5.19 The system Σ is called *uniformly observable*, with respect to the inputs, if for any two states $q_1, q_2 \in X$, such that $q_1 \neq q_2$, and any control $u(\cdot) \in \mathcal{U}$

$$y_{q_1,u}(t) \not\equiv y_{q_2,u}(t).$$

Σ is *uniformly locally observable* at $q \in X$, if there exists a neighborhood V of q, such that Σ restricted to V is uniformly observable.

Example 5.20 Example 5.12 illustrates the existence of nonlinear systems that are not uniformly observable. Another example is the unicycle, see Example 5.6, for which we observe $y_1 = x_2$ and $y_2 = x_2$. The system is observable, nevertheless, for the control $u_1(t) \equiv 0$, any two points $q = (x_{10}, x_{20}, \theta_0)^T$ and $\tilde{q} = (\tilde{x}_{10}, \tilde{x}_{20}, \tilde{\theta}_0)^T$, such that $x_{10} = \tilde{x}_{10}$ and $x_{20} = \tilde{x}_{20}$ are indistinguishable. □

Of course, linear observable systems are uniformly observable. We will describe now a class of nonlinear uniformly observable systems. Consider a single-input single-output control-affine system of the form

$$\Sigma_{\text{aff}} \ : \quad \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x), \end{array}$$

where $x \in X$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and f, g are smooth vector fields on X.

The following result is due to Gauthier and Bornard [14].

Theorem 5.21 *For the system* Σ_{aff} *we have:*

(i) *If* Σ_{aff} *is uniformly locally observable at any* $q \in X$*, then around any point of on an open and dense submanifold* X' *of* X *there exist local coordinates* $(x_1, \dots, x_n)^T$ *in which the system takes the following normal form*

$$(\mathrm{UO}) \qquad \begin{array}{rclcll} \dot{x}_1 & = & x_2 & + & ug_1(x_1), & \qquad y = x_1 \\ \dot{x}_2 & = & x_3 & + & ug_2(x_1, x_2) & \\ & \vdots & & \vdots & & \\ \dot{x}_{n-1} & = & x_n & + & ug_{n-1}(x_1, \dots, x_{n-1}) & \\ \dot{x}_n & = & f_n(x_1, \dots, x_n) & + & ug_n(x_1, \dots, x_n). & \end{array}$$

(ii) *If* Σ_{aff} *admits, locally at* q*, the form* (UO) *then it is uniformly locally observable at* q.

(iii) *A necessary and sufficient condition for* Σ_{aff} *to admit locally at* q *the normal form* (UO) *is that* $\dim \operatorname{span}\{dh, \dots, d\mathrm{L}_f^{n-1}h\}(q) = n$ *and that in a neighborhood of* q

$$[D_j, g] \subset D_j \ ,$$

for any $1 \le j \le n$*, where* $D_j = \ker\{dh, \dots, d\mathrm{L}_f^{j-1}h\}$.

5.4 Local observability: a necessary and sufficient condition

Recall that the Hermann-Krener observability rank condition gives only a sufficient condition for (strong) local observability (compare Example 5.10). Following Bartosiewicz [1] we will provide in this section a necessary and sufficient condition for local observability.

Consider a nonlinear system Σ and assume that it is analytic, that is, X is an analytic manifold, the vector fields f_u are analytic and h is an analytic map.

We start with the following simple observation.

Proposition 5.22 *The points q_1 and q_2 are indistinguishable if and only if for any $\phi \in \mathcal{H}$ we have $\phi(q_1) = \phi(q_2)$.*

Introduce now the *observation algebra* of Σ. It is the smallest subalgebra over $\mathbb{R}$ of $C^\omega(X)$, the algebra of analytic functions on X, which contains h_i and is closed under Lie derivatives with respect to $f_u,\ u \in U$. We denote it by H_A. Observe that H_A consists of all elements of H and of all constant functions.

For $x \in X$, by $\mathcal{O}_x$ we denote the algebra over $\mathbb{R}$ of germs of analytic functions at x. Denote by m_x the unique maximal ideal of $\mathcal{O}_x$. It consists of all germs that vanish at x. For $x \in X$ we define I_x to be the ideal in $\mathcal{O}_x$ generated by germs of those functions from H_A which vanish at x. Of course, $I_x \subset m_x$. The *real radical* of an ideal I in a commutative ring R is

$$\sqrt[\mathbb{R}]{I} = \{a \in R \mid a^{2m} + b_1^2 + \cdots b_k^2 \in I \text{ for some } m > 0,\ k \geq 0,\ b_1, \ldots, b_k \in R\}.$$

Clearly, the real radical is an ideal.

Theorem 5.23 *The system Σ is locally observable at x if and only if*

$$\sqrt[\mathbb{R}]{I_x} = m_x.$$

Example 5.24 We can easily see that for the system $\dot{x} = 0$, $y = x^3$ (compare Example 5.10), which is clearly locally observable at any $x \in \mathbb{R}$, we have $\sqrt[\mathbb{R}]{I_x} = m_x$ for any $x \in \mathbb{R}$, in particular, for $x = 0$. □

5.5 Generic observability properties

In this section we discuss the problem of what observability properties are shared by generic control systems. We consider C^∞-Whitney topology for smooth systems. Recall that a sequence of smooth function φ_n on a manifold X converges in C^∞-Whitney topology to a smooth function φ if there exists a compact subset $C \subset X$ such that the all derivatives $\varphi_n^{(i)}$, for $i \geq 0$, converge uniformly on C to the corresponding $\varphi^{(i)}$ and $\varphi_n \equiv \varphi$ on $X \setminus C$, for all n sufficiently large. In the case of a compact state space X, it is just the topology of C^∞ uniform convergence on X. We start by presenting results of Jakubczyk and Tchoń [28] who classified uncontrolled observed dynamics of the form

$$\Sigma : \begin{array}{rcl} \dot{x} & = & f(x) , \\ y & = & h(x) , \end{array}$$

where $x \in X$ and $y \in \mathbb{R}$, f is a smooth vector field and h is a smooth $\mathbb{R}$-valued function. Let Ξ denote the family of all systems Σ of the above form equipped with the C^∞-Whitney topology.

Theorem 5.25 *There exists an open and dense subset $\Xi_0 \subset \Xi$ such that any $\Sigma \in \Xi_0$ is locally equivalent at any $q \in X$ to one of the following normal forms.*

(i) *If $f(q) \neq 0$ then Σ is equivalent to*

$$h(x) = x_1^{r+1} + x_2 x_1^{r-1} + \cdots + x_r x_1 + \eta(x_2, \ldots, x_n), \tag{5.2}$$

$$f(x) = f_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1}, \tag{5.3}$$

where $0 \leq r \leq n$, and f_1 and η are C^∞-functions of the indicated arguments such that $f_1(0) > 0$.

(ii) *If $f(q) = 0$ then Σ is locally equivalent to*

$$h(x) = x_1 + c , \tag{5.4}$$

$$f(x) = x_2 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_{n-1}} + f_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n}, \tag{5.5}$$

where c is a constant and f_n is a C^∞-function such that $f_n(0) = 0$.

In the item (i) above, if $r = 1$ we can always take $h = x_1^2 + \eta$ while for $r = 0$ we take $h = x_1 + c$

Observe that in the case (i), for the "time-rescaled" system $\frac{dx}{d\tau} = \frac{1}{f_1(x)} f(x)$, where $d\tau = f_1(x(t))dt$, we have $x_i(\tau)$=constant, for $2 \leq i \leq n$, and thus in the new time scale

$$y(\tau) = h(x(\tau)) = \tau^{r+1} + x_2 \tau^{r-1} + \cdots x_r \tau + c,$$

where $x_2 = c_2, \ldots, x_n = c_n$ and c are constants. It follows that, firstly, responses are polynomial with respect to the new time τ, with at most r different local extreme points. Secondly, there are always initial conditions, close to q, producing $y(\tau)$ with r different local extrema.

For systems which are not generic but satisfy the observability rank condition, an analogous normal form can be established.

Theorem 5.26 *If Σ satisfies the observability rank condition at q then it is locally equivalent either to the form* (5.4)-(5.5) *if $f(q) = 0$ or, otherwise, to one of the following normal forms*

$$\begin{aligned} h(x) &= x_1^{r+1} + \phi_{r-1}x_1^{r-1} + \cdots + \phi_1 x_1 + \phi_0 + c, && (5.6) \\ f(x) &= f_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1}, && (5.7) \end{aligned}$$

where $r \geq 0$, and ϕ_i are C^∞-functions of $x_2, \ldots, x_n$, for $0 \leq i \leq r-1$, satisfying $\phi_i(0) = 0$, and f_1 is a C^∞-function such that $f_1 > 0$.

If $r = 1$ we can always take $h = x_1^2 + \phi_2$, while for $r = 0$ we take $h = x_1 + c$.

We end up this chapter by stating some results of Gauthier and Kupka devoted to the genericity of uniform observability. Consider an observed smooth control system of the form

$$\Sigma \; : \quad \begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x, u) \end{aligned}$$

where $x \in X$, $u \in U$, and $y \in Y$. Notice that we assume the output $y = h(x, u)$ to depend explicitly on the control u.

Recall that for the system Σ we define the response map, called also input-output map

$$R_\Sigma : X \times \mathcal{U} \longrightarrow \mathcal{Y},$$

which to any initial condition $x_0 \in X$ and any admissible control $u(\cdot) \in \mathcal{U}$ attaches the output of the system

$$y(t, x_0, u(\cdot)) = h(x(t, x_0, u(\cdot)), u(t)).$$

The control $u(\cdot)$ being defined on an interval $0 \in I_u \subset \mathbb{R}$, we consider the output $y(\cdot)$ on the maximal interval $I_y \subset I_u \subset \mathbb{R}$ on which it exists.

In the remaining part of this section, we will assume that the state space X is a compact manifold and $U = J^m$, where J is some compact interval of $\mathbb{R}$. We denote by Ξ the class of such systems equipped with the topology of C^∞ uniform convergence on $X \times I^m$.

For any C^k-function $w(t)$ of time we will denote $\bar{w}^k(t) = (w(t), w'(t), \ldots, w^{(k)}(t))$. For the system Σ, for any integer k and for a C^k-differentiable input $u(t)$, we define the k-prolongation of the response map as

$$R_\Sigma^k(x_0, \bar{u}^k(t)) = (y(t), y'(t), \ldots, y^{(k)}(t)) = \bar{y}^k(t),$$

that is, as the vector formed by the output and its first k derivatives with respect to time t.

For an open subset W of $\mathbb{R}^q$ (or a differential manifold), and for a C^k-differentiable function w of $I \subset \mathbb{R}$ into W, such that $0 \in I$, we denote by $j^k w$ the k-jet at $0 \in \mathbb{R}$ of w. We will denote by $J^k W$ the space of k-jets at $0 \in \mathbb{R}$ of maps from I into W. Now we consider the k-jet

$$j^k R_\Sigma : X \times J^k U \longrightarrow J^k Y$$

of the map R_Σ defined by

$$j^k R_\Sigma(x_0, j^k u) = j^k y,$$

where $j^k y = \bar{y}^k(0)$, $\bar{y}^k(t) = R^k_\Sigma(x_0, \bar{u}^k(t))$, and $u(t)$ is any C^k-control such that $\bar{u}^k(0) = j^k u$.

The following fundamental result has been proved by Gauthier and Kupka [16], [17], [18].

Theorem 5.27 *Assume $p > m$, that is, the number of outputs is greater than that of inputs. Fix a sufficiently large positive integer k.*

(i) *The set of systems Σ such that $j^k R_\Sigma(\cdot, j^k u)$ is an immersion of X into $\mathbb{R}^{p(k+1)}$, for all $j^k u \in J^k U$, contains an open dense subset of Ξ.*

(ii) *The set of systems Σ such that $j^k R_\Sigma(\cdot, j^k u)$ is an embedding of X into $\mathbb{R}^{p(k+1)}$, for all $j^k u \in J^k U$, is a residual subset of Ξ.*

(iii) *For any compact subset C of $J^k U$, the set of systems Σ such that $j^k R^k_\Sigma(\cdot, j^k u)$ is an embedding, for all $j^k u \in C$, is open dense in Ξ.*

The above result implies that, in the case $p > m$, the set of systems that are observable for all C^k inputs is residual, that is, it is a countable intersection of open dense sets. If a bound on the derivatives of the controls is given a-priori, that is $| u^{(i)}(t) | \leq M$, for some constant M and any $0 \leq i \leq k$, then this set is open dense. If the number of outputs is not greater than that of controls all statements of the above theorem are *false*.

6 Decoupling

In this section we show how static feedback allows to transform the dynamics of a nonlinear system in order to achieve desired decoupling properties. In Section 6.1 we will introduce a crucial concept of invariant distributions. In Section 6.2 we consider disturbance decoupling while in Section 6.3 we deal with input-output decoupling.

6.1 Invariant distributions

Consider a smooth nonlinear control system of the form

$$\Sigma: \quad \dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i = g_0(x) + g(x)u,$$

where $x \in X$, $u \in \mathbb{R}^m$, $g = (g_1, \dots, g_m)$ and $u = (u_1, \dots, u_m)^T$. Notice that for simplicity we denote the drift of the system by $f = g_0$.

A distribution $\mathcal{D}$ is called *invariant* for Σ if

$$[g_i, \mathcal{D}] \subset \mathcal{D}, \quad \text{for } 0 \leq i \leq m.$$

If a distribution is not invariant for Σ it may become invariant under a suitable feedback modification. A distribution $\mathcal{D}$ is called *controlled invariant* if there exists an invertible feedback of the form

$$u = \alpha(x) + \beta(x)\tilde{u}, \quad \beta(\cdot) - \text{invertible},$$

such that $\mathcal{D}$ is invariant under the feedback modified dynamics

$$\dot{x} = \tilde{g}_0(x) + \sum_{i=1}^{m} \tilde{g}_i(x)\tilde{u}_i,$$

that is,

$$[\tilde{g}_i, \mathcal{D}] \subset \mathcal{D} \ ,$$

for $0 \leq i \leq m$, where

$$\tilde{g}_o = g_o + g\alpha, \qquad \tilde{g} = g\beta.$$

Example 6.1 In the case of a linear system of the form

$$\Lambda: \quad \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$

a subspace $V \subset \mathbb{R}^n$ is said to be invariant for Λ if $AV \subset V$. We say that V is controlled invariant (or (A, B)-invariant) if there exists a linear feedback of the form $u = Fx + G\tilde{u}$ such that

$$(A + BF)V \subset V.$$

Observe that in the linear case (A, B)-invariance does not depend on G so one can take $G = \text{Id}$ or G-noninvertible.

One can check by a direct calculation that (A, B)-invariance is equivalent to

$$AV \subset V + \mathrm{Im} B. \qquad (6.1)$$

We refer to [45] for an extensive treatment of the concept of invariance in the linear case.

Put $\mathcal{G} = \mathrm{span}\,\{g_1, \ldots, g_m\}$. In the nonlinear case, controlled invariance of a distribution $\mathcal{D}$ implies the following property of *local controlled invariance* (compare (6.1))

$$[g_i, \mathcal{D}] \subset \mathcal{D} + \mathcal{G}, \quad \text{for } 0 \leq i \leq m.$$

For involutive distributions the converse holds locally under regularity assumptions, see [20],[25],[35].

Proposition 6.2 *Assume that the distributions $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{D} \cap \mathcal{G}$ are of constant rank. If $\mathcal{D}$ is involutive and locally controlled invariant then it is controlled invariant, locally at any point $x \in X$.*

6.2 Disturbance decoupling

In this Section we apply the concept of controlled invariant distributions to solve the nonlinear disturbance decoupling problem. Consider the following nonlinear system with output affected by disturbances $d = (d_1, \ldots, d_k)^T$, which are assumed to be bounded measurable $\mathbb{R}^k$-valued functions of time.

$$\Sigma_{\mathrm{dist}} : \quad \begin{array}{rcl} \dot{x} & = & g_0(x) + \sum_{i=1}^{m} g_i(x) u_i + \sum_{i=1}^{k} q_i(x) d_i = g_0(x) + g(x)u + q(x)d \\ y & = & h(x)\ , \end{array}$$

where $x \in X$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $d \in \mathbb{R}^k$. All data are smooth, i.e., $f, g_1, \ldots, g_m \in V^\infty(X)$, $h_i \in C^\infty(X)$, where $h = (h_1, \ldots, h_p)^T$, and $q_1, \ldots, q_k \in V^\infty(X)$. We denote $q = (q_1, \ldots, q_k)$ and call them disturbance vector fields.

We say that the *disturbance decoupling problem*, shortly *DDP*, is solvable, if there exists an invertible feedback of the form $u = \alpha(x) + \beta(x)\tilde{u}$ such that the output $y(t) = h(x(t))$ of the feedback modified system

$$\dot{x} = \tilde{g}_0(x) + \sum_{i=1}^{m} \tilde{g}_i(x) \tilde{u}_i + \sum_{i=1}^{k} q_i(x) d_i$$

does not depend on the disturbances $d(t)$. By the latter we mean that

$$y(t, q, \tilde{u}(\cdot), d(\cdot)) \equiv y(t, q, \tilde{u}(\cdot), \tilde{d}(\cdot)),$$

for any initial condition $q \in X$, any control $\tilde{u}(\cdot) \in \mathcal{U}$, and any disturbances $d(\cdot)$ and $\tilde{d}(\cdot)$.

Put $\mathcal{Q} = \operatorname{span}\{q_1, \ldots, q_k\}$. The following result has been proved by Isidori et al [25].

Theorem 6.3 *If DDP is solvable then there exists an involutive controlled invariant distribution $\mathcal{V}$ such that*

$$\mathcal{Q} \subset \mathcal{V} \subset \ker dh.$$

This result suggests the following approach to *DDP*. Look for the maximal controlled invariant distribution in $\ker dh$ and check whether it contains $\mathcal{Q}$. In general, however, such a maximal distribution may not exist. Moreover, even if it exists and contains the disturbance vector fields it is not necessarily true that *DDP* is solvable. On the other hand there always exists $\mathcal{V}^*$, the *maximal locally controlled invariant distribution* in $\ker dh$, which leads to the following solution of *DDP* (see [20] and [25]).

Theorem 6.4 *Assume that the distributions $\mathcal{V}^*, \mathcal{V}^* \cap \mathcal{G}$, and $\mathcal{G}$ are of constant rank. If*

$$\mathcal{Q} \subset \mathcal{V}^*,$$

then DDP is solvable, locally, around any point of X.

The structure of the decoupled system can be described as follows. Let (α, β) be an invertible feedback which locally renders the distribution $\mathcal{V}^*$ invariant (it always exists under the regularity assumptions of Theorem 6.4, see Proposition 6.2). Let $x = (x^1, x^2)$ be local coordinates, with x^1, x^2 being possibly vectors, such that $\mathcal{V}^* = \operatorname{span}\{\frac{\partial}{\partial x^2}\}$. Then the feedback modified system reads as

$$\Sigma_{\text{dist}} : \quad \begin{array}{rcl} \dot{x}^1 & = & \tilde{g}_0^1(x^1) + \tilde{g}^1(x^1)\tilde{u} \\ \dot{x}^2 & = & \tilde{g}_0^2(x^1, x^2) + \tilde{g}^2(x^1, x^2)\tilde{u} + q^2(x^1, x^2)d \\ y & = & h^1(x^1) \ , \end{array}$$

where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$. Now it is clear, compare Section 5.2, that the output $y(t)$ of the system does not depend on $d(t)$ since the latter affects the x^2-part of the system only which, in turn, is not observed by the output y.

Example 6.5 Consider the linear system with disturbances

$$\Lambda_{\text{dist}}: \quad \begin{array}{rcl} \dot{x} & = & Ax + Bu + Ed \\ y & = & Cx\ , \end{array}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p,\ d \in \mathbb{R}^k$, and d denotes the disturbances. *DDP* is solvable if and only if $\operatorname{Im} E \subset V^*$, where V^* is the largest controlled invariant subspace in ker C (compare [45]). □

Example 6.6 Consider a particle of unit mass moving on the surface of a cylinder according to a potential force given by the potential function V (see [37])

$$\begin{array}{rclrcl} \dot{q}_1 & = & p_1 & \dot{q}_2 & = & p_2 \\ \dot{p}_1 & = & -\frac{\partial V}{\partial q_1}(q_1, q_2) + u & \dot{p}_2 & = & -\frac{\partial V}{\partial q_2}(q_1, q_2) + d\ , \end{array}$$

where $(q_1, q_2, p_1, p_2) \in T(S^1 \times \mathbb{R})$. Let the output be given as $y = q_1$. We can see that $\mathcal{V}^* = \operatorname{span}\{\frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_2}\}$. Moreover, the disturbance vector field $\frac{\partial}{\partial p_2} \in \mathcal{V}^*$ and hence *DDP* is solvable by the feedback $u = \frac{\partial V}{\partial q_1}(q_1, q_2) + \tilde{u}$. □

6.3 Input-output decoupling

Consider a smooth nonlinear control affine system with outputs of the form

$$\Sigma_{\text{aff}}: \quad \begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^m u_i g_i(x) \\ y & = & h(x)\ , \end{array}$$

where $x \in X$, $u \in \mathbb{R}^m$, and $y \in R^p$.

We say that the *input-output decoupling problem* (called also *I-O decoupling problem* or *noninteracting problem*) is solvable for Σ if there exists an invertible feedback of the form $u = \alpha(x) + \beta(x)\tilde{u}$ such that the feedback modified system $\dot{x} = \tilde{f}(x) + \sum_{i=1}^m \tilde{u}_i \tilde{g}_i(x)$ with $y = h(x)$, where $\tilde{f} = f + g\alpha, \quad \tilde{g} = g\beta$, satisfies

$$y_i^{(k_i)} = \tilde{u}_i, \text{ for } 1 \le i \le p, \tag{6.2}$$

for suitable nonnegative integers k_i. Observe that we assume that the input-output map of the modified system is linear. Therefore there is no lose of generality in assuming the form (6.2) because if the transfer matrix of the input-output response is diagonal (which is the usual definition of noninteracting) we can always achieve (6.2) by applying a suitable linear feedback.

Fix an initial condition $x_0 \in X$. For each output channel we define its *relative degree* ρ_i, called also *characteristic number*, to be the smallest integer such that for any neighborhood V_{x_0} of x_0

$$L_{g_j} L_f^{\rho_i - 1} h_i(x) \neq 0,$$

for some $1 \leq j \leq m$ and for some $x \in V_{x_0}$. By $L_f^\rho h$ we will mean the vector of p smooth functions whose i-entry is $L_f^{\rho_i} h_i$.

Define the $(p \times m)$ decoupling matrix $D(x)$, denoted also by $L_g L_f^\rho h$, whose (i,j)-entry is

$$L_{g_j} L_f^{\rho_i - 1} h_i(x).$$

Theorem 6.7 *Consider a control affine system Σ_{aff}.*

(i) *The system Σ_{aff} is input-output decouplable at x_0 via an invertible feedback of the form $u = \alpha(x) + \beta(x)\tilde{u}$ if and only if*

$$\operatorname{rank} D(x_0) = p.$$

(ii) *Moreover, for the square system, i.e., $m = p$, the feedback*

$$u = -(L_g L_f^{\rho-1} h)^{-1} L_f^\rho h + (\mathrm{L}_g L_f^{\rho-1} h)^{-1} \tilde{u} \tag{6.3}$$

yields $y_i^{(k_i)} = \tilde{u}_i$, where $k_i = \rho_i$, for $1 \leq i \leq p$.

Remark 6.8 Inverting formula (6.3), we get the following expression for the new controls

$$\tilde{u}_i = L_f^{\rho_i} h_i + \sum_{j=1}^{m} u_j L_{g_j} L_f^{\rho_i - 1} h_i,$$

for $1 \leq i \leq m = p$. An analogous formula holds also in the non-square case. Indeed, if the system satisfies the decoupling condition $\operatorname{rank} D(x_0) = p$, then we can assume after a permutation of controls, if necessary, that the first p columns of the matrix $D(x_0)$ are independent. Then a decoupling feedback can be taken as

$$\begin{array}{rcll} \tilde{u}_i & = & L_f^{\rho_i} h_i + \sum_{j=1}^m u_j L_{g_j} L_f^{\rho_i - 1} h_i, & \text{for } 1 \leq i \leq p, \\ \tilde{u}_i & = & u_i, & \text{for } p+1 \leq i \leq m. \end{array} \tag{6.4}$$

Example 6.9 Consider the following rigid two-link robot manipulator or, in other words, double pendulum, (see [37], compare also Example 4.7)

$$\begin{aligned} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= -M(x^1)^{-1}(C(x^1, x^2) + k(x^1)) + M(x^1)^{-1}u \ , \end{aligned}$$

where $x^1 = \theta = (\theta_1, \theta_2)^T$, $x^2 = \dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2)^T$, $u = (u_1, u_2)^T$. The term $k(\theta)$ represents the gravitational force and the term $C(\theta, \dot{\theta})$ reflects the centripetal and Coriolis forces, and the positive definite symmetric matrix $M(x^1)$ is given by

$$\begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{pmatrix}.$$

As the outputs we take the cartesian coordinates of the endpoint

$$\begin{aligned} y_1 &= h_1(\theta_1, \theta_2) &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ y_2 &= h_2(\theta_1, \theta_2) &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2). \end{aligned}$$

By a direct computation we get $\rho_1 = \rho_2 = 2$ and $\operatorname{rank} D(x) = 2$ if and only if $l_1 l_2 \sin\theta_2 \neq 0$. Thus the system is input-output decouplable if $\theta_2 \neq k\pi$, that is, we have to exclude configurations at which the two robot arms are parallel. □

Example 6.10 Consider the unicycle (compare Examples 4.11 and 5.6) and assume that we observe the x_1-cartesian coordinate and the angle θ

$$\begin{aligned} \dot{x}_1 &= u_1 \cos\theta & y_1 &= x_1 \\ \dot{x}_2 &= u_1 \sin\theta & & \\ \dot{\theta} &= u_2 & y_2 &= \theta. \end{aligned}$$

The control u_2 has a direct impact on the second component y_2 of the output as well as, through $\cos\theta$, on the first component. Thus the system is not input-output decoupled but it can be decoupled via a static feedback. We obviously have $\rho_1 = \rho_2 = 1$ and the decoupling matrix is

$$D = \begin{pmatrix} \cos\theta & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore the system is input-output decouplable at all points such that $\theta \neq \frac{\pi}{2} + k\pi$. □

Example 6.11 Consider the same dynamics of the unicycle and suppose that this time we observe x_1 and x_2

$$\begin{array}{rclcrcl} \dot{x}_1 & = & u_1 \cos\theta & & y_1 & = & x_1 \\ \dot{x}_2 & = & u_1 \sin\theta & & y_2 & = & x_2 \\ \dot{\theta} & = & u_2. & & & & \end{array}$$

Obviously, we have $\rho_1 = \rho_2 = 1$ but this time the decoupling matrix

$$D = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix}$$

is of rank one everywhere and thus the system is not *I-O* decouplable. □

If the system is *I-O* decouplable then it is straightforward to calculate $\mathcal{V}^*$, the maximal locally controlled invariant distribution in ker dh (compare Section 6.2 and, consequently, solve the *DDP* problem.

Proposition 6.12 *Consider the system* Σ_{dist}. *Assume that the undisturbed system, that is, when* $d_i = 0$, *for* $1 \le i \le k$, *is input-output decouplable. Then*

(i) $\mathcal{V}^* = \mathcal{P}^{\perp}$, *where* $\mathcal{P} = \text{span}\,\{dL_f^j h_i,\ 1 \le i \le p,\ 0 \le j \le \rho_i - 1\}$.

(ii) *If, moreover,* $\mathcal{Q} \subset \mathcal{P}^{\perp}$ *then the DDP problem is solvable and the feedback* (6.4) *simultaneously decouples the disturbances and renders the system input-output decoupled and input-output linear.*

Example 6.13 To illustrate this result let us consider the following model of the unicycle. We suppose that the dynamics is affected by a disturbing rotation (of un unknown varying strength $d(t)$) and that we measure the angle and the square of the distance from the origin:

$$\begin{array}{rclcrcl} \dot{x}_1 & = & u_1 \cos\theta + x_2 d & & y_1 & = & x_1^2 + x_2^2 \\ \dot{x}_2 & = & u_1 \sin\theta - x_1 d & & & & \\ \dot{\theta} & = & u_2 & & y_2 & = & \theta. \end{array}$$

The decoupling matrix is

$$D = \begin{pmatrix} 2x_1 \cos\theta + 2x_2 \sin\theta & 0 \\ 0 & 1 \end{pmatrix}$$

and is of rank two at any point away from $N = \{(x_1, x_2, \theta) \mid x_1 \cos\theta + x_2 \sin\theta = 0\}$. Notice that N consists of points where the direction of the unicycle is perpendicular to the ray from the origin passing through the center of the unicycle. At points of $(\mathbb{R}^2 \times S^1) \setminus N$, the system is input-output decouplable and, moreover, $\mathcal{P} = \operatorname{span}\{x_1 dx_1 + x_2 dx_2, d\theta\}$. The vector field $q = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$ is annihilated by $\mathcal{P}$ and thus the feedback $u_1 = (2x_1 \cos\theta + 2x_2 \sin\theta)^{-1} \tilde{u}_1$ and $u_2 = \tilde{u}_2$ decouples the disturbances from the output yielding an *I-O* decoupled and *I-O* linear system expressed, in (R, φ, θ)-coordinates, where $R = r^2 = x_1^2 + x_2^2$, $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, by

$$\begin{array}{rcl} \dot{R} &=& \tilde{u}_1 \\ \dot{\varphi} &=& \frac{1}{2r^2}\tilde{u}_1 \tan(\theta - \varphi) - d \\ \dot{\theta} &=& \tilde{u}_2 \end{array} \qquad \begin{array}{rcl} y_1 &=& R \\ && \\ y_2 &=& \theta. \end{array}$$

The disturbance d does not affect the output $(y_1, y_2) = (R, \theta)$. □

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