aa **Instanton counting, Donaldson invariants, line bundles on moduli spaces of sheaves on rational surfaces**

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(0) Introduction

Donaldson invariants: C^∞-inv. of compact 4-manifolds
For X proj. surface: intersection number $\int_M \mu(L)^d$ on moduli space $M_H^X(c_1, c_2)$ of H-stable rk 2 sheaves on X

Nekrasov partition function $Z(\epsilon_1, \epsilon_2, a, \Lambda)$:
generating function of "equiv. Donaldson inv. of $\mathbf{A}^2$"
equivariant intersection number of moduli of instantons $M(n)$.

X proj. toric surface $\Longrightarrow$ X glued together from $\mathbf{A}^2$'s

Aim A: Compute Donaldson invariants of X in terms of Nekrasov partition function

K-theory Nekrasov partition function $Z^K(\epsilon_1, \epsilon_2, a, \Lambda)$:
generating function for $ch(H^*(M(n), \mathcal{O}))$
"K-theoretic Donaldson invariants of $\mathbf{A}^2$"

$$\begin{matrix} L \text{ line bundle} \\ \text{on } X \end{matrix} \mapsto \begin{matrix} \overline{L} := \mathcal{O}(\mu(L)) \text{ Donaldson l. b.} \\ \text{on } M_H^X(c_1, c_2) \end{matrix}$$

Aim B: Compute $\chi(M_X^H(c_1, c_2), \overline{L})$ in terms of Z^K

Note: D-inv $\phi^H_{X,c_1}(c_1(L)^d) = \int_{M_H^X(c_1,c_2)} c_1(\overline{L})^d$
Riemann-Roch $\Longrightarrow \chi(M, \overline{L}) = \phi^H_{X,c_1}(c_1(\overline{L})^d)/d! + l.o.t$

Motivation:

(1) Nekrasov partition function is closely related to relation Seiberg Witten-invariants $\longleftrightarrow$ Donaldson invariants

(2) Formula can be viewed as analogue of topological vertex formula

(3) $\chi(M, \overline{L})$ should be K-theoretic Donaldson invariants (still not constructed).
Want to understand analogues of all basic properties of Donaldson-invariants

(1) Nekrasov Partition function

Instanton moduli space $\mathbf{P}^2 = \mathbf{A}^2 \cup l_\infty$,

$$M(n) := \left\{ \begin{matrix} \text{framed coh. sheaves } (E,\phi) \text{ on } \mathbf{P}^2 \\ rk(E) = 2, c_2(E) = n, \phi : E|_{l_\infty} \simeq \mathcal{O}_{l_\infty}^{\oplus 2} \end{matrix} \right\}$$

smooth quasiproj, dim $4n$

Torus Action:

$\mathbf{C}^* \times \mathbf{C}^*$ acts on $(\mathbf{P}^2, l_\infty)$: $(e^{\epsilon_1}, e^{\epsilon_2})(z_1, z_2) = (e^{\epsilon_1} z_1, e^{\epsilon_2} z_2)$

$\mathbf{C}^*$ acts on $M(n)$ by change of framing

$e^a(E,\phi) = (E, diag(e^a, e^{-a}) \circ \phi)$

Fixpoint set of $(\mathbf{C}^*)^3$ is finite:

$$\left\{ (I_{Z_1} \oplus I_{Z_2}, id) \;\middle|\; Z_i \in Hilb^{n_i}(\mathbf{A}^2, 0), \text{ ideal gen. by monomials} \right\}$$

Let X variety over $\mathbf{C}$ with action of
$T = (\mathbf{C}^*)^k = \{(e^{w_1}, \ldots, e^{w_k})\}$ and $X^T = \{p_1, \ldots, p_e\}$
Equiv. cohom. $H_T^*(X)$: module over $H_T^*(pt) = \mathbf{C}[w_1, \ldots, w_k]$

Equiv. integration: X compact, $\widetilde{\alpha}$ equiv. lift of $\alpha \in H^*(X)$

$$\int_X \alpha = \sum_{p_i} \frac{\widetilde{\alpha}|_{p_i}}{c_{top}(T_{p_i}X)}\Big|_{w_1=\ldots=w_k=0}$$

Note $\widetilde{\alpha}|_{p_i}$, $c_{top}(T_{p_i}X) \in \mathbf{C}[w_1, \ldots, w_k]$

Nekrasov Partition function
$M(n)$ with action of $(\mathbf{C}^*)^3 = \{(e^{\epsilon_1}, e^{\epsilon_2}, e^a)\}$

$$Z^{inst}(\epsilon_1, \epsilon_2, a, \Lambda) = \text{"} \sum_{n \geq 0} \Lambda^{4n} \int_{M(n)} 1\text{"} \in \mathbf{C}(\epsilon_1, \epsilon_2, a)[[\Lambda]]$$

$Z(\epsilon_1, \epsilon_2, a, \Lambda) = Z^{inst} \cdot Z^{per}$

Nekrasov conjecture

(Nekrasov-Okounkov, Yoshioka-Nakajima)

(1) $Z = \exp(\frac{F(\epsilon_1,\epsilon_2,a,\Lambda)}{\epsilon_1\epsilon_2})$, F regular at $\epsilon_1 = \epsilon_2 = 0$

(2) $F_0 = F|_{\epsilon_1=\epsilon_2=0}$ is Seiberg-Witten prepotential (periods of SW-curve, a family of elliptic curves).

K-theory Nekrasov Partition function

$$Z_K^{inst}(\epsilon_1,\epsilon_2,a,\Lambda) = \sum_{n\geq 0} \Lambda^{4n} e^{-(\epsilon_1+\epsilon_2)n} \sum_i (-1)^i ch(H^i(M(n),\mathcal{O})) \in \mathbf{C}(\epsilon_1,\epsilon_2,a)[[\Lambda]]$$

in equivariant K-theory. Eigenspaces of are finite dim.

$Z_K = Z_K^{inst} \cdot Z_K^{per}$

(Yoshioka-Nakajima): Similar result for Z_K

Same statement, different family of elliptic curves

Know also next two orders in ϵ_1, ϵ_2 of F and F_K

(2) Review of Donaldson invariants for alg. surfaces

(X, H) projective surface

$$M_H^X(c_1, c_2) = \{H\text{-stable rank 2 sheaves}\}$$

$\mathbf{E} \to X \times M$ universal sheaf

$$L \in H_2(X) \mapsto \mu(L) = (2c_2(\mathbf{E}) - \frac{1}{2}c_1(\mathbf{E})^2)/L \in H^2(M)$$

$$\phi^X_{c_1,H}(\exp(L)) = \sum_n \int_{M_H^X(c_1,n)} exp(\mu(L))\Lambda^{dim(M)}$$

• $p_g(X) > 0$: independent of H
• $p_g(X) = 0$: depends on H via system of walls and chambers in ample cone $\mathbf{C}_X$

Walls: $\xi \in H^2(X, \mathbf{Z})$ defines wall of type (c_1, c_2) if
$\xi \equiv c_1 \mod 2H^2(X, \mathbf{Z})$ and $4c_2 - c_1^2 + \xi^2 \geq 0$
Wall $W^\xi := \{H \in \mathbf{C}_X \mid H \cdot \xi = 0\}$

Chambers=connected components of $C_X \setminus$ walls
$M_X^H(c_1, c_2)$ and invariants constant on chambers, change when H crosses wall (i.e. $H_- \to H_+$ with $H_-\xi < 0 < H_+\xi$)

Kotschick-Morgan conj.: wallcrossing for Donaldson inv. is polynomial in $\langle \xi, L \rangle$, L^2, coefficients depend only on c_2, ξ^2 and topology of X

Using K-M conjecture [G] determined gen. function for wallcrossing in terms of modular forms
See also Moore-Witten, Marino-Moore

(3) Donaldson inv. and $\chi(M,\overline{L})$ versus Nekrasov part. fctn

X smooth toric surface, has $\mathbf{C}^* \times \mathbf{C}^* = \{(e^{\epsilon_1}, e^{\epsilon_2})\}$-action
fixpoints $\{p_1, \ldots, p_e\}$, $w(x_i), w(y_i)$ weights of action on $T_{p_i}X$
Fix F nef with $M_F^X(c_1, c_2) = \emptyset$ for all c_2. For H ample:

Donaldson invariants:

$$\phi^X_{c_1,H}(\mathrm{exp}(L)) = \sum_\xi \lim_{\epsilon_1,\epsilon_2\to 0} \mathop{\mathrm{Coeff}}_{t^0} \left(-\frac{t}{\Lambda} \prod_{i=1}^{e} Z\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{\frac{L}{2}|_{p_i}}\Big)\right)$$

Here $\xi \in H^2(X,\mathbf{Z})$ with $\xi \equiv c_1 \mod 2$ and $\xi H > 0 > \xi F$.

Holomorphic Euler Characteristic:

$$\sum_n \chi(M^X_H(c_1,n), \overline{L})\Lambda^{dimM}$$

$$= \sum_\xi \lim_{\epsilon_1,\epsilon_2\to 0} \mathop{\mathrm{Coeff}}_{(e^t)^0} \left(-\frac{1}{\Lambda} \prod_{i=1}^{e} Z_K\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{\frac{2L-K}{4}|_{p_i}}\Big)\right)$$

(3) Method of proof

X toric surface $\Longrightarrow M_H^X(c_1,c_2)$ and Don. inv. depend on chamber of H. In one chamber moduli space is empty $\Longrightarrow$ everything determined by wallcrossing

Aim: Relate wallcrossing to partition function.
Let ξ define wall W^ξ. When H crosses wall, sheaves E in

$$0 \to I_Z((\xi + c_1)/2) \to E \to I_W((-\xi + c_1)/2) \to 0, \quad (Z,W) \in X^{[l]} \times X^{[m]}$$

$(l + m = 4c_2 - c_1^2 + \xi^2)$ are replaced by extensions the other way round

Geometrically: Start with $M_{H_-}^X(c_1,c_2)$.
Successively for $l = 0, 1 \ldots, 4c_2 - c_1^2 + \xi^2$:
- blowup bundle $\mathbf{P}(\mathrm{Ext}^1(I_W, I_Z(\xi)))$ over $X^{[l]} \times X^{[m]}$,
- blow down exceptional divisor D to $\mathbf{P}(\mathrm{Ext}^1(I_Z, I_W(-\xi)))$.

Finally arrive at $M_{H_+}^X(c_1,c_2)$.

Fix l, m, let $p : D \to X^{[l]} \times X^{[m]}$ projection

$$\left[\text{wallcrossing for D-inv}\right] = \int_D (Ap^*(B)) = \int_{X^{[l]}\times X^{[m]}} (p_* A) B$$
$$\left[\text{wallcrossing for } \chi(\overline{L})\right] = \chi(D, A' p^*(B')) = \chi(X^{[l]} \times X^{[m]}, p_*(A')B')$$

Now apply Bott formula on $X^{[l]} \times X^{[m]}$:
Action of $\mathbf{C}^* \times \mathbf{C}^*$ on X lifts to $X^{[l]} \times X^{[m]}$

$$\bigcup_{l,m} (X^{[l]} \times X^{[m]})^{(\mathbf{C}^*)^2} = \bigcup_{n_1,\ldots,n_e} M(n_1)^{(\mathbf{C}^*)^3} \times \ldots \times M(n_e)^{(\mathbf{C}^*)^3}$$

Show: contribution for both sides is the same at every fixpoint

$T_{(Z,W)} X^{[n]} \times X^{[m]} = \mathrm{Ext}^1(I_Z, I_Z) \oplus \mathrm{Ext}^1(I_W, I_W)$
$T_{(Z,W)} M(n) = \mathrm{Ext}^1(I_Z, I_Z) \oplus \mathrm{Ext}^1(I_W, I_W) \oplus \mathrm{Ext}^1(I_Z, I_W)e^{2a} \oplus \mathrm{Ext}^1(I_W, I_Z)e^{-2a}$

(4) Explicit formulas in modular forms and elliptic functions

Develop $F = \epsilon_1\epsilon_2 \log Z$, $F_K = \epsilon_1\epsilon_2 \log Z_K$:

$$F(\epsilon_1,\epsilon_2,a,\Lambda) = F_0 + (\epsilon_1+\epsilon_2)H + \epsilon_1\epsilon_2 F_1 + (\epsilon_1+\epsilon_2)^2 G + h.o.t$$

Similarly for F_K. Then

$$\prod_{i=1}^{e} Z\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{\frac{L}{2}|_{p_i}}\Big) = \exp\left(\sum_{i=1}^{e} \frac{1}{w(x_i)w(y_i)} F\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{\frac{L}{2}|_{p_i}}\Big)\right)$$

$$= \exp\left(\sum_i \frac{1}{w(x_i)w(y_i)} \frac{\partial^2 F_0}{(\partial a)^2}(t/2,\Lambda)\frac{(\xi|_{p_i})^2}{8} + \ldots\right)$$

$$= \exp\Big(\frac{\partial^2 F_0}{(\partial a)^2}\frac{\xi^2}{8} - \frac{\partial^2 F_0}{\partial \log\Lambda \partial a}\frac{\langle \xi, L\rangle}{4} + \frac{\partial^2 F_0}{(\log\Lambda)^2}\frac{\langle L, L\rangle}{4} + \ldots\Big)$$

by Bott formula on X. Similarly for Z_K.

Put $\tau := -\frac{1}{2\pi i}\frac{\partial^2 F_0}{(\partial a)^2}$ period of SW elliptic curve.

$h := -\frac{1}{8}\frac{\partial^2 F_0}{\partial \log\Lambda \partial a}$, $Q := \frac{1}{16}\frac{\partial^2 F_0}{(\partial \log\Lambda)^2}$, similarly for h_K, Q_K.

$$q := e^{\pi i \tau}, \quad \theta_{\mu\nu}(z,\tau) := \sum_{n\in Z} (-1)^{(n+\frac{\mu}{2})\nu} q^{(n+\frac{\mu}{2})^2} e^{2\pi i (n+\frac{\mu}{2})z}, \quad \theta_{\mu\nu}(\tau) := \theta_{\mu\nu}(0,\tau), \quad \mu,\nu = 0,1$$

Donaldson invariants

$$\phi^X_{c_1,H}(\exp(L)) = \sum_\xi \pm \operatorname*{Coeff}_{q^0} \left(q^{-(\xi/2)^2} \exp\left(\langle \xi, 2L\rangle h\Lambda + (2L)^2 Q\Lambda^2\right) \theta_{01}(\tau)^{\sigma(X)} B \right)$$

.

Holomorphic Euler characteristic

$$\sum_{n\in\mathbf{Z}} \chi(M^X_H(c_1,n), \overline{L})\Lambda^{dim(M)}$$

$$= \sum_\xi \pm \operatorname*{Coeff}_{q^0} \left(q^{-(\xi/2)^2} \exp\left(\langle \xi, 2L-K\rangle h_K + (2L-K)^2 Q_K\right) \theta_{01}(\tau)^{\sigma(X)} B_K \right)$$

$$h_K(\Lambda) = 2\pi i \left(i \frac{\theta_{11}(\bullet,\tau)}{\theta_{01}(\bullet,\tau)} \right)^{-1} = h\Lambda + O(\Lambda^3)$$

$$Q_K = \log\left(\frac{\theta_{01}(\frac{h_K}{2\pi i},\tau)}{\theta_{01}(\tau)} \right) = Q\Lambda^2 + O(\Lambda^4)$$

$$B_K = \frac{q\frac{d}{dq}\sqrt{(1-\Lambda^2)^2 + \frac{4\Lambda^2\theta_{00}(\tau/2)^4}{\theta_{10}(\tau/2)^4}}}{4\Lambda^2\theta_{10}(\tau/2)^2} = B + O(\Lambda^2)$$