Sections of line bundles on moduli spaces of sheaves on surfaces and strange duality

Lothar Göttsche
based on
(1) joint work with H.Nakajima, K.Yoshioka,
(2) help by D.Zagier

Vancouver, Feb. 2, 2009

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H-semistable: $\frac{\chi(\mathcal{F}\otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{F})} \le \frac{\chi(\mathcal{E}\otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{E})}$ for all $0 \neq \mathcal{F} \subset \mathcal{E}$, $n \gg 0$
H-slope stable: $\frac{c_1(\mathcal{F})H}{\mathrm{rk}(\mathcal{F})} < \frac{c_1(\mathcal{E})H}{\mathrm{rk}(\mathcal{E})}$

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For simplicity assume universal sheaf $\mathcal{E}$ on $X \times M$
i.e. $\mathcal{E}|_{X\times[E]} = E$ for all $[E] \in M$

$K^0(X)$:=Grothendieck group of vector bundles
Let $c \in K^0(X)$ class of $E \in M$.

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Definition

Let $v \in K^0(X)$ with $\chi(X, c \otimes v) = 0$ (write $v \in c^\perp$)
The *determinant bundle* for v is

$$\lambda(v) := \det(Rp_*(\mathcal{E} \otimes q^*(v))^{-1} \in Pic(M)$$

$\lambda : c^\perp \to Pic(M)$ is homomorphism.

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Let $L \in Pic(X)$. Assume Lc_1 even. Put

$$v(L) := \mathcal{O}_X - L^{-1} + k\mathcal{O}_{pt} \in c^\perp$$

The Donaldson line bundle for L is $\widetilde{L} := \lambda(v(L))$.

Definition

The K-theoretic Donaldson invariant for L is $\chi(M, \widetilde{L})$.

Generating function:

$$\sum_{c_2} \chi(M_X^H(c_1, c_2), \widetilde{L})\Lambda^d \in \mathbb{Z}[[\Lambda]].$$

$d = 4c_2 - c_1^2 - 3 = expdim(M)$

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Standard Donaldson invariant

$$\int_M c_1(\widetilde{L})^d = \lim_{n\to\infty} \frac{d!}{n^d}\chi(M, \widetilde{nL})$$

by Riemann-Roch
(K-th Don. invariants are refinement of standard Don. inv.).

$M_X^H(c_1, c_2)$ depends on H:
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- Relate result to Le Potier's strange duality conjecture

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Definition

Let $C_X \subset H^2(X, \mathbb{R})$ be the ample cone.
$\xi \in H^2(X, \mathbb{Z})$ defines wall of type (c_1, c_2) if

1. $\xi \equiv c_1 \mod 2H^2(X, \mathbb{Z})$
2. $4c_2 - c_1^2 + \xi^2 \geq 0$
3. the wall $W^\xi := \{H \in C_X \mid H \cdot \xi = 0\}$ is nonempty.

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Chambers=connected components of $C_X \setminus$ walls
$M_X^H(c_1, c_2)$ and invariants constant on chambers, change when H crosses wall (i.e. $H_- \to H_+$ with $H_-\xi < 0 < H_+\xi$)

Definition

Let ξ define a wall of type (c_1, c_2). Put $d := 4c_2 - c_1^2 - 3$
The wallcrossing term is
$\Delta^X_{\xi,d}(L) := \chi(M_X^{H_+}(c_1, c_2)), \widetilde{L}) - \chi(M_X^{H_-}(c_1, c_2)), \widetilde{L})$.

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First aim: give a generating function for the wallcrossing terms in terms of elliptic functions.

Theta functions:

$\theta_{ab}(h) = \sum_{n\equiv a\ (2)} q^{n^2}(i^b y)^n,\ a,b\in\{0,1\},\ y=e^{h/2},\ q=e^{\pi i\tau/4}$

$\theta_{ab} := \theta_{ab}(0),\ u := -\frac{\theta_{00}^2}{\theta_{10}^2} - \frac{\theta_{10}^2}{\theta_{00}^2}, \quad \Lambda := \frac{\theta_{11}(h)}{\theta_{01}(h)}$

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Theorem

Write

$$q^{-\xi^2}y^{\xi(L-K_X)}\left(\frac{\theta_{01}(h)}{\theta_{01}}\right)^{(L-K_X)^2}\theta_{01}^{\sigma(X)}q\frac{du}{dq}\frac{dh}{d\Lambda}=\sum_{d\in\mathbb{Z}_{\geq 0}}f_d(q)\Lambda^d.$$

$$f_d(q)\in\mathbb{Q}((q))$$

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Then

$$\Delta^X_{\xi,d}(L) := \chi(M^{H_+}_X(c_1,c_2),\widetilde{L}) - \chi(M^{H_-}_X(c_1,c_2),\widetilde{L}) = \pm\,\mathrm{Coeff}_{q^0} f_d(q)$$

Generating function

$$\Delta^X_\xi(L) := \sum_d \Delta^X_{\xi,d}(L)\Lambda^d = \sum_d \mathrm{Coeff}_{q^0} f_d(q)\Lambda^d \in \mathbb{Z}[[\Lambda]]$$

Remark

(bad news) $\sum_n \chi(M_X^H(c_1,n),\widetilde{L})\Lambda^d$ *has wallcrossing for all* W^ξ *with* $\xi \in c_1 + 2H^2(X,\mathbb{Z})$*: Walls are everywhere dense in* C_X*.*

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Corollary

(good news) Let ξ *class of type* (c_1, c_2)

1. $\Delta_\xi^X(L) \in \mathbb{Z}[\Lambda]$ *(a polynomial!)*
2. *If* $|\xi(L - K_X)| + 1 \leq -\xi^2$ *then* $\Delta_\xi^X(L) = 0$
   "Most walls do not contribute at all".

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Corollary

$\sum_n \chi(M_X^H(c_1, n), \widetilde{L})\Lambda^d$ *independent of* H *up to adding a polynomial.*

Main ingredient in proof of wallcrossing formula:
Nekrasov partition function.

Instanton moduli space:
$M(n) = \{(E, \phi) \mid E \text{ rk } 2, \text{ sheaf on } \mathbb{P}^2 \text{ with } c_2(E) = n, \phi : E|_{l_\infty} \simeq \mathcal{O}^{\oplus 2}\}$

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Torus action: $\mathbb{C}^* \times \mathbb{C}^*$ acts on $(\mathbb{P}^2, l_\infty)$:
$(t_1, t_2)(z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2 z_2)$.
Extra $\mathbb{C}^*$ acts by $s(E, \phi) = (E, diag(s^{-1}, s) \circ \phi)$.

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Fixpoints: $M(n)^{(\mathbb{C}^*)^3} = \left\{(I_{Z_1} \oplus I_{Z_2}), id) \mid Z_i \in Hilb^{n_i}(\mathbb{A}^2, 0) \text{ monomial}\right\}$

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Character: Let V vector space with $(\mathbb{C}^*)^3$ action. $\Longrightarrow V = \bigoplus_i V_{M_i}$
V_{M_i} eigenspace with eigenvalue M_i Laurent monomial in t_1, t_2, s.
The Character of V is $ch(V) := \sum_i dim(V_{M_i}) M_i$

The Nekrasov partition function is given by

$$Z^{inst}(\epsilon_1,\epsilon_2,a,\Lambda) := \sum_{n\geq 0}\left(\frac{\Lambda^4}{t_1 t_2}\right)^n ch(H^0(M(n),\mathcal{O}))|_{t_1=e^{\epsilon_1},t_2=e^{\epsilon_2},s=e^a}$$

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$Z = Z^{inst} Z^{pert}$, where Z^{pert} is explicit function of $\epsilon_1, \epsilon_2, a, \Lambda$.

Nekrasov Conjecture (Nekrasov-Okounkov, Nakajima-Yoshioka, Braverman-Etingof):

1. $F(\epsilon_1, \epsilon_2, a, \Lambda) = \epsilon_1 \epsilon_2 log(Z)$, F regular at $\epsilon_1, \epsilon_2 = 0$
2. $F_0(a, \Lambda) = F|_{\epsilon_1=\epsilon_2=0}$ can be expressed in terms of elliptic functions.

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Wallcrossing: replace sheaves in extensions

$$0 \to I_{Z_1}((c_1+\xi)/2) \to E \to I_{Z_2}((c_1-\xi)/2) \to 0, \qquad Z_i \in Hilb^{n_i}(X)$$

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$\Longrightarrow$ change from $M_X^{H_-}(c_1,c_2)$ to $M_X^{H_+}(c_1,c_2)$ by series of flips with centers $Hilb^{n_1}(X)\times Hilb^{n_2}(X)$.
Compute $\Delta^X_{\xi,d}(L)$ as inters. numbers on the $Hilb^{n_1}(X)\times Hilb^{n_2}(X)$.

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Compute $\Delta^X_{\xi,d}(L)$ as inters. numbers on the $Hilb^{n_1}(X) \times Hilb^{n_2}(X)$.
Reduce to case X is toric. Use localization to compute intersection number in terms of weights at fixpoints.
Get product of Nekrasov partition functions over the fixpoints of the action on X with $\epsilon_1, \epsilon_2, a$ replaced by weights of the action.

Let X 1-conn. alg. surf., let $c, v \in K^0(X)$ with $\chi(c \otimes v) = 0$.
Assume $H^2(E \otimes F) = 0$ for all $[E] \in M(c)$, $[F] \in M(v)$.

$$\Theta := \left\{(E, F) \in M(c) \times M(v) \mid h^0(E \otimes F) \neq 0\right\}$$

Assume Θ is zero set of $\sigma \in H^0(M(c) \times M(v), \lambda(v) \boxtimes \lambda(c))$
$\Longrightarrow$ Duality morphism $D : H^0(M(c), \lambda(v))^\vee \to H^0(M(v), \lambda(c))$

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Conjecture/Question

1. *Is $D : H^0(M(c), \lambda(v))^\vee \to H^0(M(v), \lambda(c))$ an isomorphism? (strong strange duality)*
2. *Is $\chi(M(c), \lambda(v)) = \chi(M(v), \lambda(c))$? (weak strange duality)*

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For curves: rank/level duality conj. of Beauville, Donagi-Tu.
Weak version is Corollary of Verlinde formula.
Strong version proved by Belkale, Marian-Oprea.

For simplicity $c_1 = 0$.
$c(n) = \mathcal{O}^{\oplus 2} - n\mathcal{O}_{pt}$ class of $E \in M_X^H(0, n)$
Let $L \in Pic(X)$, $v(L) = \mathcal{O}_X - L^{-1} + k\mathcal{O}_{pt} \in c(n)^{\perp}$
Put $\theta := \lambda(\mathcal{O}_X), \eta := \lambda(-\mathcal{O}_{pt}) \in Pic(M(v(L)))$
$\Longrightarrow \lambda(c(n)) = \theta^{\otimes 2} \otimes \eta^{\otimes n}$.

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Strange duality: $\chi(M_X^H(0, n), \widetilde{L}) = \chi(M(v(L)), \lambda(c(n)))$
$\Longrightarrow \sum_n \chi(M_X^H(0, n), \widetilde{L})t^n = \sum_n \chi(M(v(L)), \theta^{\otimes 2} \otimes \eta^{\otimes n})t^n \in \mathbb{Q}(t)$

Remark

There is natural morphism $\pi : M(v(L)) \to |L|, \mathcal{F} \mapsto supp(\mathcal{F})$
General fibre over $[C]$ is $Pic^{g-1}(C)$, $(g = g(C))$.
Restriction of θ to $Pic^{g-1}(C)$ is the theta bundle, $\eta = \pi^*(\mathcal{O}(1))$.

Assuming strange duality, get

$$\begin{aligned} \chi\big(M_X^H(0, c_2), \widetilde{L}\big) &= \chi(M(v(L)), \theta^{\otimes 2} \otimes \pi^*(\mathcal{O}(c_2))) \\ &= \chi(|L|, \pi_*(\theta^{\otimes 2}) \otimes \mathcal{O}(c_2)) \end{aligned}$$

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Over general $C \in |L|$ have
$rank(\pi_*\theta^2) = H^0(Pic^{g-1}(C), 2\theta) = 2^g$
Assume best of all worlds: $\pi_*\theta^2 = \bigoplus_{i=1}^{2^g} \mathcal{O}(-a_i)$, all $a_i < \chi(X, L)$. Then
$\sum_{c_2 \geq 0} \chi\big(M_X^H(0, c_2), \widetilde{L}\big) t^{c_2} =$
$\sum_{c_2 \geq 0} H^0\big(|L|, \bigoplus_{i=1}^{2^g} \mathcal{O}(-a_i + c_2)\big) t^{c_2} = \frac{\sum_{i=1}^{2^g} t^{a_i}}{(1-t)^{dim|L|+1}}$.

Let X be a rational surface. Possibly after blowing up X there is an H_0 (on boundary of ample cone) with $\chi(M_X^{H_0}(c_1, c_2), \widetilde{L}) = 0$ for all c_2. $\Longrightarrow$ Everything is determined by wallcrossing:

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Problem: H_0 is not ample. The sum will be infinite (infinitely many summands nonzero). Need arguments about elliptic functions/modular forms to carry it out.

Let X rational ruled surface. F fibre. Let $G \in \frac{1}{2}H^2(X,\mathbb{Z})$ with $G^2 = 0$, $FG = 1$. Let $L \in Pic(X)$. Write $L = nF + sG$, $s \in \mathbb{Z}$, $n \in \frac{1}{2}\mathbb{Z}$
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Theorem

1. *Let* $c_1 = 0, F$, *then*

$$1 + \sum_{c_2>0} \chi\big(M_X^H(c_1,c_2),\widetilde{L}\big)t^{c_2} \simeq \begin{cases} \frac{1}{(1-t)^{n+1}} & s=0 \\ \frac{1}{(1-t)^{2n+2}} & s=1 \\ \frac{1}{2}\frac{(1+t)^n+(-1)^{c_1\cdot L/2}(1-t)^n}{(1-t)^{3n+3}} & s=2 \end{cases}$$

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3. *Explicit formulas for $L = nF + sG$, $s \leq 7$.*

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$$\sum_n \chi(M^H_{\widehat{X}}(c_1, n), \widetilde{L})t^n = \sum_n \chi(M^H_X(c_1, n), \widetilde{L})t^n$$

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One also needs higher order blowup formulas for $\widetilde{L - mE}$.
These involve an analogue of the point class.

Theorem

$$1+\sum_{c_2>0}\chi\big(M_{\mathbb{P}^2}(0,c_2),\widetilde{nH}\big)t^{c_2}=\begin{cases}\frac{1}{(1-t)^3} & n=1\\ \frac{1}{(1-t)^6} & n=2\\ \frac{1+t^2}{(1-t)^{10}} & n=3\\ \frac{1+6t^2+t^3}{(1-t)^{15}} & n=4\\ \frac{1+21t^2+20t^3+21t^4+t^6}{(1-t)^{21}} & n=5\end{cases}$$

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(Danila determined other side of strange duality for H, $2H$, $3H$ and checked strange duality in this case for small c_2.)

For rational surfaces the generating functions are rational:

Theorem

Let X rational surface, H ample, $L, c_1 \in Pic(X)$ with Lc_1 even. There are (computable) $P^X_{L,c_1}(t) \in \mathbb{Q}[t]$, $l \in \mathbb{Z}$ s.th.

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Conjecture

If $h^i(X, L) = 0$ for $i > 0$ and general $C \in |L|$ is nonsing. genus g, then $l = \chi(X, L)$ and $P^X_{L,c_1}(t)$ has nonnegative coefficients and $P^X_{L,c_1}(1) = 2^g$.

(True in all cases I checked)

Again X rational surface H ample on X, $L \in Pic(X)$
Write

$$\chi^H_{X,c_1}(L,\Lambda) := \sum_{n\geq 0} \chi(M^H_X(c_1,n),\widetilde{L})\Lambda^d \in \mathbb{Z}[\Lambda, 1/(1-\Lambda^4)]$$

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What is the explanation of the Le Potier dual statement?
Fourier-Mukai transform for $M(v) \to |L|$?

Example proof: $X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = nF$. To show

$$\chi(nF) := 1 + \sum_{c_2} \chi(M_X^H(F, c_2), \widetilde{nF})\Lambda^{4c_2} = \frac{1}{(1-\Lambda^4)^{n+1}}.$$

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$$\begin{aligned}
&(1-\Lambda^4)\chi((n+1)F) - \chi(nF) \\
&= -\sum_{k\geq 0}(1-\Lambda^4)\left(\Delta^X_{(2k+1)F}((n+1)F) - \Delta^X_{(2k+1)F}(nF)\right) \\
&= -\sum_{k\geq 0}\operatorname{Coeff}_{q^0}\left[y^{4k+2}((1-\Lambda^4)Q^{4n+12} - Q^{4n+8}R\right] \\
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$Q = \frac{\theta_{01}(h)}{\theta_{01}}$, $y = e^{h/2}$, $\Lambda = \frac{\theta_{11}(h)}{\theta_{01}(h)}$.

Enough to show: $\operatorname{Coeff}_{q^0}\left[\frac{1}{y^2 - y^{-2}}((1-\Lambda^4)Q^{4n+8} - Q^{4n+4}) \times R\right] = 0$

Turns out to follow from $\theta_{00}^4 = \theta_{10}^4 + \theta_{01}^4$ [Jacobi]

Know $Q^4 \in \mathbb{Q}[[q^2\Lambda^2, q^4]]$. Will show

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For $y = \pm 1$ this is $0 = 0$. For $y = \pm i$ it is

$$\theta_{00}^4 = \theta_{10}^4 + \theta_{01}^4 \qquad \text{[Jacobi]}$$

General formula, involves analogue of point class.
$U \in \widetilde{K}^0(M^H_X(c_1, c_2))$ universal. $U = (p_{2!}((\mathcal{E} \cdot \mathcal{E}^\vee) \cdot p_1^*(\mathcal{O}_{\{x\}}))$??

Theorem

There are universal polynomials $P_m(x,t)$, such that

$$\sum_n \chi(M^H_{\widehat{X}}(c_1, n), \widetilde{L - mE})t^n = \sum_n \chi(M^H_X(c_1, n), \widetilde{L} \otimes P_m(U, t))t^n$$

$\chi(M, \widetilde{L} \otimes U^k t^n) := \chi(M, \widetilde{L} \otimes U^{\otimes k})t^n$
Let $M := 2\frac{\theta_{01}^4}{\theta_{10}^2\theta_{00}^2}\frac{\theta_{10}(h)^2\theta_{00}^2(h)}{\theta_{01}^4(h)}$, $\Lambda = \frac{\theta_{11}(h)}{\theta_{01}(h)}$. Then

$$\frac{\theta_{01}((m+1)h)\theta_{01}^{(m+1)^2-1}}{\theta_{01}(h)^{(m+1)^2}} = P_m(M^2, \Lambda^4)$$

$$P_0 = 1,\ P_1 = (1-t),\ P_2 = (1-t)^2 - tx,\ P_3 = (1-t)^4 - tx^2$$

.