Refined curve counting and tropical geometry

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The whole series is about joint works with Vivek Shende, Florian Block, Benjamin Kikwai, Franziska Schroeter

Lecture 4: Refined Broccoli invariants

Welschinger invariants: Let S real algebraic surface
until now only considered **totally real** Welschinger invariants, i.e. the curves are required to pass through real points, now impose point conditions given by pairs of complex conjugate points
consider irreducible curves of genus 0, i.e. $\delta = g(L)$
P configuration of r real points in S and s pairs of complex conjugate points with $(r + 2s) = dim|L| - g(L)$
Welschinger invariants: $W^0_{(S,L),r,s} = \sum_C (-1)^{s(C)}$
sum is over all real nodal rational curves C in $|L|$ though P
$s(C) = \#\{\text{isolated nodes of } C\}$

In case of $\mathbb{P}^2$:
$W^0_{d,r,s} = \sum_C (-1)^{s(C)}$, $r + 2s = 3d - 1$.

What I say holds for toric surfaces, for simplicity restrict to $\mathbb{P}^2$
One can count $W^0_{d,r,s}$ using graphs, similar to tropical curves considered below, the Welschinger curves, counting them with suitable multiplicities.
They are not so well behaved, so Broccoli curves were introduced
They compute the same invariants, but are better behaved
drawbacks:

- they are quite complicated. There are many different vertex types depending on the parity of the weights of the edges
- these vertices all are counted with different multiplicities
- in case $s = 0$ the Broccoli curves do not specialise to the tropical curves

Introduce refined Broccoli curves (which contain Broccoli curves as subset)
and refined Broccoli invariants $N^{0,trop}_{d,r,s}(y)$ in $\mathbb{Z}[y, y^{-1}]$

- simpler than the Broccoli curves and invariants
- only two vertex types with corresponding multiplicities
- in case $s = 0$ the refined Broccoli curves specialise to the tropical curves
- $N^{0,trop}_{d,r,s}(-1) = W^0_{d,r,s}$, $N^{0,trop}_{d,r,0}(y) =\mathrm{N}^{0,trop}_{d,\binom{d-1}{2}}(y)$ (irred. refined Severi degree).

plane Broccoli curve of degree d**=plane irreducible rational tropical curve of degree d**:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
4. Γ has d unbounded edges in each of the directions $(1,1)$, $(-1,0)$, $(0,-1)$

But the point conditions are different!

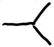
Curves Γ counted in $N^{0,trop}_{d,r,s}(y)$ pass through r (thin) points p_i and s (fat) points P_i, where passing through a fat points means, that P_i lies on a vertex of Σ

Again we count simple (in particular trivalent) refined Broccoli curves through r thin and s fat points. Count them again with a vertex multiplicity ($m(v)$ =Mikhalkin multiplicity)

Two kinds of vertices:

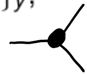
Standard vertex: $M(v) = [m(v)]_y$

$[n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$



Fat vertex: $M(v) = \{m(v)\}_y$,

$\{n\}_y = \frac{y^{n/2} + y^{-n/2}}{y^{1/2} + y^{-1/2}}$



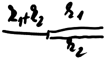
$$M(\Gamma) = \prod_{v \text{ vertex}} M(v), \quad N^{0,trop}_{d,r,s}(y) = \sum_{\Gamma} M(\Gamma)$$

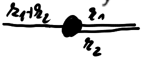
$/2^{\#\text{double ends}}$

sum over all genus 0 degree d simple tropical curves through r thin and s fat points.

Some examples

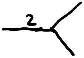
$m(v) = 0$ | $M(v) = 0$ | $M(v) = \frac{2}{y^{1/2}+y^{-1/2}}$



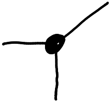


$m(v) = 1$ | $M(v) = 1$ | $M(v) = 1$

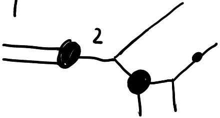
m(v)=2 | $M(v)=y^{1/2} + y^{-1/2}$ | $M(v) = \frac{y+y^{-1}}{y^{1/2}+y^{-1/2}}$



$N^{0,trop}_{1,0,1}(y) = 1$



$N^{0,trop}_{2,1,2}(y) = 1$



Theorem

1. $N^{0,trop}_{d,r,s}(y) \in \mathbb{Z}[y, y^{-1}]$
2. $N^{0,trop}_{d,r,s}(y)$ *is a tropical invariant (independent of (generic) position of points)*
3. $N^{0,trop}_{d,r,s}(-1) = W^{0}_{d,r,s}$
4. $N^{0,trop}_{d,r,0}(y) = N^{0,trop}_{d,\binom{d-1}{2}}(y)$ *(irred. refined Severi degree)*

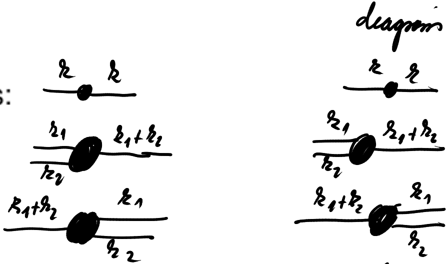
Similar results hold for general convex lattice polygons.
For (3): if Γ is not Broccoli in the old sense, then $M(\Gamma)|_{y=-1} = 0$, and if Γ is Broccoli, then $M(\Gamma))|_{y=-1}$ is the Broccoli multiplicity (although the contribution of the vertices is different). For many vertices $M(v)|_{y=-1} = 0$ or $M(v)|_{y=-1} = \infty$, this is why nonrefined Broccoli multiplicities have to be more complicated.

Floor diagrams:

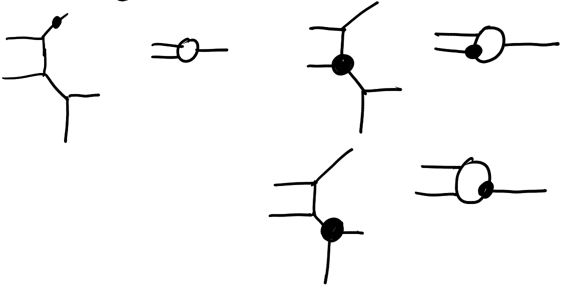
again curves through horizontally stretched configuration of points have floor decomposition
A horizontal edge of C is called an **escalator**
A connected component of closure of complement of escalators in Γ is called a **floor**.
The following properties hold:

1. Every floor and every escalator contains precisely one marked point.
2. Only the escalators can have weights different from 1
3. any vertex v has multiplicity $m(v) = 1$, unless it is adjacent to am escalator e, in which case the multiplicity is $m(v) = w(e)$.

Escalators:



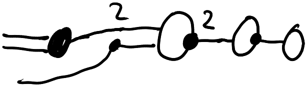
Floors:



To count Refined Broccoli we can just count floor diagrams

Description of floor diagrams

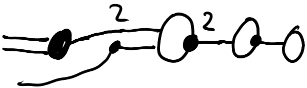
1. Every bounded edge connects a black (either inside white of just black) and white vertex
2. Every unbounded edge connects to something black
3. white vertices v can have several incoming and outgoing edges
   $div(v) = \sum_{e-incoming} w(e) - \sum_{e-outgoing} w(e) = 1$



$s=3$
$r=2$
$d=3$
$g=0$

Counting of floor diagrams

$$M(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y \prod_{v \text{ fat vertex}} \{w(v)\}_y \Big/ 2^{\#\text{double ends}}$$



$$\frac{1}{y^{1/2}+y^{-1/2}} \cdot (y^{1/2}+y^{-1/2}) \frac{y+y^{-1}}{y^{1/2}+y^{-1/2}} (y^{1/2}+y^{-1/2})$$
$$= (y+y^{-1})$$

By definition:

Proposition

$$N^{0,trop}_{d,r,s}(y) = \sum_{\text{connected genus } 0 \ (r,s)\text{-floor diagrams } \Lambda \text{ of degree } d} M(\Lambda)$$

As for the usual Severi degrees gives rise to a Caporaso Harris recursion by removing the vertices starting from the left.

Again the floor diagrams can be viewed as Feynman diagrams for certain operators on the same Fock space as before.

H deformed Heisenberg algebra gen. by $a_n, b_n, \quad n \in \mathbb{Z}$
commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \quad [a_n, b_m] = [n]_y \delta_{n,-m}, \quad [n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$$

Fock space: F generated by **creation operators** a_{-n}, b_{-n}
acting on vacuum vector $v_\emptyset$
Basis paramtr. by pairs of partitions
$\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$, $\nu = (1^{\nu_1}, 2^{\nu_2}, \ldots)$
$a_\mu := \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} := \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$, similarly for b_ν, $b_{-\nu}$
$v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$ basis for F

inner product $\langle v_\emptyset | v_\emptyset \rangle = 1$; a_n, b_n adjoint to a_{-n}, b_{-n}

$$C(t) = \sum_{k_1,k_2>0} \frac{1}{y^{1/2}+y^{-1/2}} (b_{k_1+k_2} b_{-k_1} b_{-k_2} + b_{k_1} b_{k_2} b_{-k_1-k_2})$$

$$+ t \sum_{k>0} \frac{y^{k/2}+y^{-k/2}}{y^{1/2}+y^{-1/2}} \left(\sum_{\|\mu\|=\|\nu\|-1+k} b_k a_\nu a_{-\mu} + \sum_{\|\mu\|=\|\nu\|-1-k} a_\nu a_{-\mu} b_{-k} \right)$$

$$H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$$

$\|\mu\| := \sum_i i\mu_i$; sum includes $\mu = \emptyset$

Generating function

$$\sum_{d\geq 0} \sum_{\delta\geq 0} \frac{t^d Q^s q^{3d-1-2s}}{s!(3d-1-2s)!} N^{0,trop}_{d,r,s}(y)$$

$$= \operatorname{Res}_{z=0} \left[\log \left(\langle v_\emptyset, \exp(Qz^2 C(t/z^3)) \cdot \exp(qzH(t/z^3)) \exp(a_{-1}) v_\emptyset \rangle \right) \right]$$

We are working on the definition and invariance and also recursion formulas for higher genus possibly reducible Broccoli invariants. There are some partial results and some interesting conjectures. I cannot believe I will have the time to say anything about it.