Moduli spaces and Modular Forms

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Aim of this talk: Relate generating functions of invariants of moduli spaces in algebraic geometry to modular forms

What do all these words mean?

Generating functions: Assume $(a_n)_{\geq 0}$ are interesting numbers. Their generating function is

$$f(t) := \sum_{n\geq 0} a_n t^n$$

Want a nice closed formula for $f(t)$

Example

p_n = number of Partitions of n. $p_0 = 1, p_1 = 1, p_2 = 2, p_3 = 3$ $((3), (2,1), (1,1,1))$

$$\sum_{n\geq 0} p_n t^n = \prod_{k\geq 1} \frac{1}{1-t^k}.$$

Study (projective) algebraic Varieties:
Projective space: $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$, $v \sim \lambda v$ for $\lambda \in \mathbb{C}$

Algebraic variety: Let $F_1, \dots, F_r \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous polynomials

$$Z(F_1, \dots, F_r) = \{(p_0, \dots, p_n) \mid F_i(p_0, \dots, p_n) = 0, \ i = 1, \dots, r\}$$

A variety X is called smooth if it is a complex manifold.
Dimension is the dimension as complex manifold, i.e. a curve (dimension 1) is a Riemann surface. Varieties can be singular.

Moduli space: A variety M parametrizing interesting objects

Example

Elliptic curve$=$ (E curve of genus 1, point $0 \in E$)
Then $E \simeq E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$

$$E_\tau \simeq E_{\frac{a\tau+b}{c\tau+d}}, \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z})$$

$\Longrightarrow M_{1,1} = \{\text{Moduli space of elliptic curves}\} = \mathbb{H}/Sl(2,\mathbb{Z})$
Compactify: $\overline{M}_{1,1} = M_{1,1} \cup \infty$

Modular forms: "Functions" (sections of line bundles) on moduli space $\overline{M}_{1,1}$ of elliptic curves

Definition

Modular form of weight k on $Sl(2,\mathbb{Z})$: holomorphic function $f : \mathbb{H} \to \mathbb{C}$ s.th

1. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z})$.
2. f is "holomorphic at ∞":

$$f(\tau) = \sum_{n\geq 0} a_n q^n \quad q = e^{2\pi i\tau}, \ a_n \in \mathbb{C}$$

f is a cusp form, if also $a_0 = 0$
Similar definition for modular forms on subgroups of $Sl(2,\mathbb{Z})$ of finite index, maybe also with character

Example

Eisenstein series: $G_k(\tau) = -\frac{B_k}{2k} + \sum_{n\geq 1}\Big(\sum_{d|n} d^{k-1}\Big)q^n, \quad k > 2$ even

modular form of weight k

Discriminant: $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}$ cusp form of weight 12

Ring of Modular forms: closed under multiplication $M_* = \mathbb{C}[G_4, G_6]$

Generalizations:

1. Quasimod. forms: $QM_* = \mathbb{C}[G_2, G_4, G_6]$ closed under $D = q\frac{d}{dq}$
2. Mock modular forms: holom. parts of real analytic modular forms

Why should we care about modular forms?

1. Come up in many different parts of mathematics and physics: q-development is generating function for interesting things
2. There are very few modular forms ($\Longrightarrow$ relations between interesting numbers from different fields)

Topological invariants: Betti numbers $b_i(M) = dimH^i(M)$, Euler number $e(M) = \sum(-1)^i b_i(M)$,
intersection numbers

$$\int_{[M]} \alpha_1 \cup \ldots \cup \alpha_s, \qquad \alpha_i \in H^{n_i}(M)$$

Examples of the last: Donaldson invariants, Donaldson-Thomas invariants, Gromov-Witten invariants

What are Cohomology, Betti numbers (extremely roughly):
$b_i(X) =$"number of holes of codim i"
$=$"essentially different i-codim 'submanifolds' of X"
If $\alpha_i \in H^i(X)$ are represented by submanifolds V_i then

$$\int_{[M]} \alpha_1 \cup \ldots \cup \alpha_s = \text{"\#intersection points } \bigcap_i V_i\text{"}$$

Can also thing of deRham cohomology: $H^i(X) = Ker(d\big|\Omega^i_X)/d(\Omega^{i-1}_X)$
Then intersection number is $\int_{[M]} \alpha_1 \wedge \ldots \wedge \alpha_s$.

Generating functions of invariants of moduli spaces

Moduli spaces M_n depending on $n \geq 0$, find a nice formula for the invariants of all at the same time

Example

$\mathbb{P}^n =$ moduli space of 1-dim subvectorspaces in $\mathbb{C}^{n+1}$

$e(\mathbb{P}^n) = n + 1$, thus $\sum_n e(\mathbb{P}^n)t^n = \frac{1}{(1-t)^2}$

In general would think: hard enough to compute for one M_n

But: often easier for generating functions: relations between different M_n give differential equation for generating function

Aim: Compute generating functions of invariants of moduli spaces M_n depending on $n \geq 0$. Show they are modular forms

Too simple example: Euler numbers of symmetric powes:
S smooth surface, symm. grp $G(n)$ acts on S^n permuting factors
$S^{(n)} = S^n/G(n)$ symm. power: (singular) projective variety
Moduli space of n points on S with multipl.: points of $S^{(n)}$ = sets $\{(p_1, n_1), \ldots, (p_r, n_r)\}$, $p_i \in S$ distinct, $n_i > 0$, $\sum n_i = n$
Betti numbers: $b_i(X) := \dim H^i(X, \mathbb{Q})$, $p(X, z) := \sum_{i=0}^{\dim X} b_i(X) z^i$,
$e(X) = \sum_{i=0}^{n} (-1)^i b_i(X) = p(X, -1)$ Euler number

Theorem (MacDonald formula)

$$\sum_{n\geq 0} p(S^{(n)}, z) t^n = \frac{(1+zt)^{b_1(S)}(1+z^3t)^{b_3(S)}}{(1-t)^{b_0(S)}(1-z^2t)^{b_2(S)}(1-z^4t)^{b_4(S)}}$$

Corollary

$$\sum_{n\geq 0} e(S^{(n)}) t^n = \frac{1}{(1-t)^{e(S)}}$$

$S^{[n]}$ =Hilbert scheme of n points on S, different moduli of n pts on S
Points of $S^{[n]}$: $\{(p_1, \mathcal{O}_1), \ldots, (p_r, \mathcal{O}_r)\}$, $p_i \in S$, $\mathcal{O}_i$ quotient of dim. n_i of holom. fcts near p_i, $S^{[n]}$ is nonsingular
Morphism: $\omega_n : S^{[n]} \to S^{(n)}, \{(p_i, \mathcal{O}_i)\} \mapsto \{(p_i, n_i)\}$
Study this map, its fibres ...

Theorem (G)

$$\sum_{n\geq 0} p(S^{[n]}, z)t^n = \prod_{k\geq 1} \frac{(1+z^{2k-1}t^k)^{b_1(S)}(1+z^{2k+1}t^k)^{b_3(S)}}{(1-z^{2k-2}t^k)^{b_0(S)}(1-z^{2k}t^k)^{b_2(S)}(1-z^{2k+2}t^k)^{b_4(S)}}$$

Corollary

$$\sum_{n\geq 0} e(S^{[n]})q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)^{e(S)}} = \left(\frac{q}{\Delta(\tau)}\right)^{e(S)/24}$$

Later developments:

1. One of the motivating examples of S-duality conjecture of Vafa-Witten: Generating fct for Euler numbers of moduli spaces of stable sheaves should be modular forms. (Explain later)
2. Vafa-Witten also say: formula means: $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$ is irreducible representation of Heisenberg algebra.
   Essentially this means: $\exists$ very nice way to make $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$ out of $H^*(S, \mathbb{Q})$. Proved by Nakajima, Groijnowsky Lehn, Lehn-Sorger, ... Carlsson-Okounkov: rich algebraic structure on $\bigoplus_n H^*(S^{[n]}, \mathbb{Q})$

Generalization to dimension 3 X smooth 3-fold. Cheah proves

$$\sum_{n\geq 0} e(X^{[n]})q^n = \prod_{k\geq 1}\left(\frac{1}{(1-q^k)^k}\right)^{e(X)}.$$

This is related to Donaldson-Thomas invariants (Maulik-Nekrasov-Okounkov-Pandharipande, Behrend-Fantechi, ...).

S proj. alg. surface. This means S has embedding in some $\mathbb{P}^N$
Usually do not care about embedding (as long as it exists)
Let H ample on S ($=$hyperpl. section of embed. $S \subset \mathbb{P}^n$).
Fixing H essentially means fixing embedding of S in $\mathbb{P}^n$

A **vector bundle** of rank r on S "is" $\pi : E \to S$, such that all fibres are complex vector spaces of rank r
The Chern classes $c_1(E) \in H^2(S, \mathbb{Z})$, $c_2(E) \in H^4(S, \mathbb{Z})$ measure how different E is from $\mathbb{C}^r \times S$.

Fix $c_1 \in H^2(S,\mathbb{Z})$, $c_2 \in H^4(S,\mathbb{Z})$ Chern classes

$$M := M_S^H(c_1, c_2)$$
$$= \text{moduli space of } H\text{-stable rk 2 sheaves on } S \text{ with } c_1, c_2$$

sheaf="vector bundle with singularities"
H-stable: "all subsheaves of $\mathcal{E}$ are small"; depends on H

$M \supset N$ =stable vector bundles (open subset).
Look at generating functions:

$$Z_{c_1}^{S,H} := \sum_n e(M_S^H(c_1, n))q^{n-c_1^2/4}$$
$$Y_{c_1}^{S,H} := \sum_n e(N_S^H(c_1, n))q^{n-c_1^2/4}$$

S-duality conj. (Vafa-Witten): $Z_{c_1}^{S,H}$, $Y_{c_1}^{S,H}$ are (almost) modular forms

Theorem (Compatibilty results (Yoshioka, G, Qin-Li-Wang ...))

1. $Z_{c_1}^{S,H} = \left(\frac{q}{\Delta(\tau)}\right)^{e(S)/12} Y_{c_1}^{S,H}$
2. *(Blowup formula:) $\widehat{S} \to S$ blowup of S in a point (replace p by a $\mathbb{P}^1$).*

$$Z_{c_1}^{\widehat{S},H} = \theta(\tau)\left(\frac{q}{\Delta(\tau)}\right)^{1/12} Z_{c_1}^{S,H}, \quad \theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

(for both formulas relate difference of both sides to Hilbert scheme of points)

Special surfaces:

K3 surfaces: 1-connected proj. surface with nowhere vanishing holomorphic 2 form, e.g quartic in P^3

Theorem (G-Huybrechts, Yoshioka,...)

Let S be a K3 surface, if c_1 is not divisible by 2 in $H^2(S,\mathbb{Z})$, then $e(M) = e(S^{[\dim(M)/2]})$

For the proof relate the moduli space to Hilbert schemes, in fact they are shown to be diffeomorphic

Projective plane:

$H(n) = \#\{$ quadrat. forms $ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{Z}$
with $b^2 - ac = -n\}/iso$
$G_{3/2}(\tau) := \sum_{n \geq 0} H(n)q^n = \frac{-1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + \ldots$ Mock modular form

Theorem (Klyachko)

$e(N_{\mathbb{P}^2}(H, n)) = 3H(4n - 1)$, *thus*

$$Y_H^{\mathbb{P}^2} = \frac{3}{2}\Big(G_{3/2}(\tau/4) - G_{3/2}((\tau + 2)/4)\Big)$$

$N_{\mathbb{P}^2}(H, n)$ has a $\mathbb{C}^*$ action, $e(N) = \#$fixpoints

Wallcrossing:

Let S rational surf., e.g. (multiple) blowup of $\mathbb{P}^2$
$M_X^H(c_1, c_2)$ depends on $H \in C_S = \{H \in \mathbb{H}^2(S, \mathbb{R}) \mid H^2 > 0\}$
There are walls (=hyperplanes) dividing C_X into chambers
$M_X^H(c_1, c_2)$ const. on chambers, changes when H crosses wall
Change: replace $\mathbb{P}^k$ bundles over $S^{[n]}$ by $\mathbb{P}^l$-bundles
everything understood in terms of Hilbert schemes

Theorem (G)

Let S rational surface, H ample on S $\Longrightarrow$
$Z_{c_1}^{S,H}$ is a mock modular form.

Donaldson invariants: $\mathbb{C}^\infty$ invariants of X diff. 4-manifold def. using moduli spaces of asd connections (solutions of PDE)

Now S proj. alg. surface. D-invariants can be defined using moduli spaces $M = M_S^H(c_1, c_2)$ of stable sheaves on S

$\mathcal{E}/S \times M$ universal sheaf (i.e. restriction to $S \times [E]$ is E)
Let $L \in H_2(S, \mathbb{Q})$. Put $\mu(L) := 4c_2(\mathcal{E}) - c_1(\mathcal{E})^2/L \in H^2(M, \mathbb{Q})$.
Donaldson invariant

$$\Phi^H_{X,c_1}(L^d) = \int_X \mu(L)^d, \quad d = dim(M)$$

Generating function: $\Phi^H_{X,c_1}(e^{Lz}) = \sum_d \Phi^H_{X,c_1}(L^d)\frac{z^d}{d!}$

Rational surfaces: Seen: $M_S^H(c_1, c_2)$ subject to wallcrossing

G,G-Nakajima-Yoshioka: Generating function for wallcrossing of Donaldson invariants in terms of modular forms $\Longrightarrow$ generating function for invariants for rational surfaces in terms of modular forms

Case of $\mathbb{P}^2$:

Theorem (G, G-Nakajima-Yoshioka)

$$\Phi_H^{\mathbb{P}^2}(\exp(Hz)) = \sum_{0<n\leq m} \operatorname*{Coeff}_{q^0} \left[\frac{q^{\frac{4m^2-(2n-1)^2}{8}}}{\sqrt{-1}^{6n-2m+5}} \exp\left((n-1/2)hz + Tz^2\right)\theta_{01}^9 h^3 \right]$$

$u := -\frac{\theta_{00}^4+\theta_{10}^4}{\theta_{00}^2\theta_{10}^2}$, $h := \frac{2\sqrt{-1}}{\theta_{00}\theta_{10}}$, $T := -h^2 G_2 - \frac{u}{6}$,
$\theta_{00} := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}$, $\theta_{10} = \sum_{n\in\mathbb{Z}} q^{\frac{(n+\frac{1}{2})^2}{2}}$

G uses (unproven) Kotschick-Morgan conjecture: wallcrossing term should only depend on topology

G-N-Y uses instanton counting (maybe see Bruzzo's talk)

A different formula in terms of modular forms was proposed (based on physics arguments) by Moore-Witten. Ono-Malmendier recently proved both formulas are equal

Let S proj. surface, L holom. line bundle on S, $s : S \to L$ section
Zero set $Z(s)$ is (possibly singular) curve on S
denote $|L|$ set of all such curves
A curve $C \subset S$ is rational, if image of a map $\mathbb{P}^1 \to X$

K3 surfaces: Let S a $K3$ surface (e.g. quartic in $\mathbb{P}^3$)
L lb on S, s.th all $Z(s) \in |L|$ are irreducible (not union of curves)

Theorem (Yau-Zaslow, Beauville, Fantechi-G-van Straten)

 # rational curves in $|L|$ (with multipl.) depends only on $c_1(L)^2 \in 2\mathbb{Z}$
Denote it $n_{c_1(L)^2/2}$. Then

$$\sum_{k\in\mathbb{Z}} n_k q^k = \frac{1}{\Delta(\tau)}$$

Proof again consist in relating this to Hilbert schemes of points

Let S proj. surface, L line bundle on S
$a_\delta(S,L) = \#\delta$-nodal curves in $|L|$ through $h^0(L)-1-\delta$-points on S

Conjecture (G)

1. $\exists$ *polynomials* $T_\delta(x,y,z,w)$ *s.th* $\forall S$, *all sufficiently ample* L
$a_\delta(S,L) = T_\delta(h^0(L), \chi(\mathcal{O}_S), c_1(L)K_S, K_S^2)$
2. $\exists$ *power series* $B_1, B_2 \in \mathbb{Z}[[q]]$ *s.th.*

$$\sum_{\delta\geq 0} T_\delta(x,y,z,w)(DG_2)^\delta = \frac{(DG_2/q)^x B_1^z B_2^w}{(\Delta(\tau)D^2G_2/q^2)^{y/2}}$$

A line bundle L on S is ample if $c_1(L)$ is the hyperplane section of a projective embedding. Then $c_1(L)^2 > 0$, $c_1(L)C > 0$ for all curves in S
Sufficiently ample: these numbers are large enough wrt δ
($h^0(L) =$ dim(space of sections of L), $K_S =$zero set of holom.2-form, $\chi(\mathcal{O}_S) = 1 - h^0(\Omega^1) + h^0(\Omega^2)$)